CONSTRUCTIVE REPRESENTATION THEORY FOR THE FEYNMAN OPERATOR CALCULUS

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Abstract. In this paper, we survey recent progress on the constructive theory of the Feynman operator calculus. We first develop an operator version of the Henstock-Kurzweil integral, and a new Hilbert space that allows us to construct the elementary path integral in the manner originally envisioned by Feynman. After developing our time-ordered operator theory we extend a few of the important theorems of semigroup theory, including the Hille-Yosida theorem. As an application, we unify and extend the theory of time-dependent parabolic and hyperbolic evolution equations. We then develop a general perturbation theory and use it to prove that all theories generated by semigroups are asympotic in the operator-valued sense of Poincaré. This allows us to provide a general theory for the interaction representation of relativistic quantum theory. We then show that our theory can be reformulated as a physically motivated sum over paths, and use this version to extend the Feynman path integral to include more general interactions. Our approach is independent of the space of continuous functions and thus makes the question of the existence of a measure more of a natural expectation than a death blow to the foundations for the Feynman integral.

1. **Introduction**

In elementary quantum theory, the (simplest) problem is to solve

$$
\begin{aligned}
&i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} - \frac{1}{2m}\Delta\psi(\mathbf{x},t) = 0,\ \psi(\mathbf{x},s) = \delta(\mathbf{x}-\mathbf{y}),\\
&\psi(\mathbf{x},t) = K\left[\mathbf{x},\ \ t;\ \mathbf{y},\ \ s\right] = \left[\frac{2\pi i\hbar(t-s)}{m}\right]^{-3/2}\exp\left[\frac{im}{2\hbar}\frac{\left|\mathbf{x}-\mathbf{y}\right|^2}{(t-s)}\right].
\end{aligned}
\tag{1.1}
$$

In his formulation of quantum theory, Feynman wrote the solution to equation (1.1) as

$$K\left[\mathbf{x},\, t;\; \mathbf{y},\, s\right] = \int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x} \mathcal{D}\mathbf{x}(\tau) \exp\left\{\tfrac{im}{2\hbar}\int_s^t \left|\tfrac{d\mathbf{x}}{dt}\right|^2 d\tau\right\}, \tag{1.2}$$

where

$$\begin{aligned} &\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x} \mathcal{D}\mathbf{x}(\tau) \exp\left\{\tfrac{im}{2\hbar}\int_s^t \left|\tfrac{d\mathbf{x}}{dt}\right|^2 d\tau\right\} =: \\ &\lim_{N\to\infty}\left[\tfrac{m}{2\pi i\hbar\varepsilon(N)}\right]^{3N/2} \int_{\mathbf{R}^3} \prod_{j=1}^{N} d\mathbf{x}_j \exp\left\{\tfrac{i}{\hbar}\sum_{j=1}^{N}\left[\tfrac{m}{2\varepsilon(N)}\left(\mathbf{x}_j-\mathbf{x}_{j-1}\right)^2\right]\right\}, \end{aligned} \tag{1.3}$$

with $\varepsilon(N) = (t-s)/N$.

Problems

. Equation (1.3) represents an attempt to define an integral on the space of continuous paths with values in $\mathbf{R}^3$ (e.g., $\mathbf{C}\left([s,\ t]:\ \mathbf{R}^3\right)$).

- The kernel $K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ and $\delta(\mathbf{x})$, are not in $L^2[\mathbf{R}^n]$, the standard space for quantum theory.
- The kernel $K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ cannot be used to define a measure.

If we treat $K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ as the kernel for an operator acting on good initial data, then a partial solution has been obtained by a number of workers. (See the recent book by Johnson and Lapidus [JL] for references to all the important contributions in this direction.)

Question

. Is there a separable Hilbert space containing $\mathbf{K}\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ & $\delta(\mathbf{x})$? This is required if the above limit is to make sense.

Since the position and momentum, $\mathbf{x}, \mathbf{p}$ are canonically conjugate variables (e.g., Fourier transform pairs), any Hilbert space containing $\mathbf{K}\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ & $\delta(\mathbf{x})$ must also allow the convolution and Fourier transform as bounded operators. This requirement is necessary if we are to make sense of equation (1.3), and have a representation space for basic quantum theory. (Recall, it is precisely the realization that one cannot associate a countably additive measure with the Feynman integral that has led many to question much of the mathematical integrity of modern physics, where this integral is routinely used.)

Purpose

. The purpose of this review is to provide a survey of recent progress on the constructive theory for the Feynman operator calculus (see Gill and Zachary [GZ]). (The theory is constructive in that operators acting at different times actually commute.) The work in [GZ] was primarily written for researchers concerned with the theoretical and/or mathematical foundations for quantum field theory. (A major objective was to prove two important conjectures of Dyson for quantum electrodynamics, namely that in general, we can only expect the perturbation expansion to be asymptotic, and that the ultraviolet divergence is caused by a violation of the Heisenberg uncertainty relation at each point in time.)

In that paper, it was argued that a correct formulation and representation theory for the Feynman time-ordered operator calculus should at least have the following desirable features:

- It should provide a transparent generalization of current analytic methods without sacrificing the physically intuitive and computationally useful ideas of Feynman.
- It should provide a clear approach to some of the mathematical problems of relativistic quantum theory.
- It should explain the connection with path integrals.

This paper is written for the larger research community including applied and pure mathematics, biology, chemistry, engineering and physics. With this in mind, and in order to make the paper self contained, we have provided a number of results and ideas that may not be normal fare. We assume the standard mathematics background of an aggressive graduate student in engineering or science, and have provided proofs for all nonstandard material.

Summary

. In Section 1.1 we introduce the Henstock-Kurzweil integral (HK-integral). This integral is easier to understand (and learn) compared to the Lebesgue or Bochner integrals, and provides useful variants of the same theorems that have made those integrals so important. Furthermore, it arises from a simple (transparent) generalization of the Riemann integral that was taught in elementary calculus. Its usefulness in the construction of Feynman path

integrals was first shown by Henstock [HS], and has been further explored in the book by Muldowney [MD].

In Section 1.2, We construct a new Hilbert space that contains the class of HK-integrable functions. In order to show that this space has all the properties required to provide a complete answer to our question and for our later use, Section 1.3, is devoted to a substantial review of operator theory, including some recently published results and some new results on operator extensions that have not appeared elsewhere. As an application, we show that the Fourier transform and the convolution operator have bounded extensions to our new Hilbert space. In Section 1.4 we review the basics of semigroup theory and in Section 1.5, we apply our results to provide a rigorous proof that the elementary Feynman integral exists on the new Hilbert space.

In Section 2, we construct the continuous tensor product Hilbert space of von Neumann, which we use to construct our version of Feynman's film. In Section 3 we define what we mean by time ordering, prove our fundamental theorem on the existence of time-ordered integrals and extend basic semigroup theory to the time-ordered setting, providing among other results, a time-ordered version of the Hille-Yosida Theorem. In Section 4 we construct time-ordered evolution operators and prove that they have all the expected properties. As an application, we unify and extend the theory of time-dependent parabolic and hyperbolic evolution equations.

In Section 5 we define what is meant by the phase "asymptotic in the sense of Poincaré" for operators. We then develop a general perturbation theory and use it to prove that all theories generated by semigroups are asympotic in the operator-valued sense of Poincaré. This result allows us to extend the Dyson expansion and provide a general theory for the interaction representation of relativistic quantum theory.

In Section 6 we return to the Feynman path integral. First, we show that our theory can be reformulated as a physically motivated sum over paths. We use this version to extend the Feynman path integral in a very general manner and prove a generalized version of the well-known Feynman-Kac theorem. The theory is independent of the space of continuous functions and hence makes the question of measure more of a desire than a requirement. (Whenever a measure exists, our theory can be easily restricted to the space of continuous paths.)

1.1. **Henstock-Kurzweil integral.**

. The standard university analysis courses tend to produce a natural bias and unease concerning the use of finitely additive set functions as a basis for the general theory of integration (despite the efforts of Alexandroff [AX], Bochner [BO], Blackwell and Dubins [BD], Dunford and Schwartz [DS], de Finetti [DFN] and Yosida and Hewitt [YH]).

Without denying an important place for countable additivity, Blackwell and Dubins, and Dubins and Prikry (See [BD], [DUK], and [DU]) argues

forcefully for the intrinsic advantages in using finite additivity in the basic axioms of probability theory. (The penetrating analysis of the foundations of probability theory by de Finetti [DFN] also supports this position.) In a very interesting paper, [DU] Dubins shows that the Wiener process has a number of "cousins", related processes all with the same finite dimensional distributions as the Wiener process. For example, there is one cousin with polynomial paths and another with piece-wise linear paths. Since the Wiener measure is unique, these cousins must necessarily have finitely additive limiting distributions.

In this section, we give an introduction to the class of HK-integrable functions, while providing a generalization to the operator-valued case. The integral is well defined for operator-valued functions that may not be separably valued (where both the Bochner and Pettis integrals are undefined). Loosely speaking, one uses a version of the Riemann integral with the interior points chosen first, while the size of the base rectangle around any interior point is determined by an arbitrary positive function defined at that point. This integral was discovered independently by Henstock [HS] and Kurzweil [KW]. In order to make the conceptual and technical simplicity of the HK-integral available to all, we prove all except the elementary or well-known results. Let $\mathcal{H}$ be a separable Hilbert space and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let $[a,b] \subset \mathbb{R}$ and for each $t \in [a,b]$, let $A(t) \in L(\mathcal{H})$ be a given family of operators.

Definition 1. *Let $\delta(t)$ map $[a,b] \to (0,\infty)$, and let $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$, where $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$. We call $\mathbf{P}$ a HK-partition for δ (or HK-partition when δ is understood) provided that for $0 \leqslant i \leqslant n-1$, $t_i, t_{i+1} \in (\tau_{i+1} - \delta(\tau_{i+1}), \tau_{i+1} + \delta(\tau_{i+1}))$.*

Lemma 2. *(Cousins Lemma) If $\delta(t)$ is a mapping of $[a,b] \to (0,\infty)$ then a HK-partition exists for δ.*

Lemma 3. *Let $\delta_1(t)$ and $\delta_2(t)$ map $[a,b] \to (0,\infty)$, and suppose that $\delta_1(t) \leqslant \delta_2(t)$. Then, if $\mathbf{P}$ is a HK-partition for $\delta_1(t)$, it is also one for $\delta_2(t)$.*

Definition 4. *The family $A(t)$, $t \in [a,b]$, is said to have a (uniform) HK-integral if there is an operator $Q[a,b]$ in $L(\mathcal{H})$ such that, for each $\varepsilon > 0$, there exists a function δ from $[a,b] \to (0,\infty)$ such that, whenever $\mathbf{P}$ is a HK-partition for δ, then*

$$\left\| \sum\nolimits_{i=1}^{n} \Delta t_i A(\tau_i) - Q[a,b] \right\| < \varepsilon.$$

In this case, we write

$$Q[a,b] = (HK) \int_a^b A(t)dt.$$

Theorem 5. *For $t \in [a,b]$, suppose the operators $A_1(t)$ and $A_2(t)$ both have HK-integrals, then so does their sum and*

$$(HK) \int_a^b [A_1(t) + A_2(t)]dt = (HK) \int_a^b A_1(t)dt + (HK) \int_a^b A_2(t)dt.$$

Theorem 6. *Suppose $\{A_k(t) \mid k \in \mathbb{N}\}$ is a family of operator-valued functions in $L[\mathcal{H}]$, converging uniformly to $A(t)$ on $[a,b]$, and $A_k(t)$ has a HK-integral $Q_k[a,b]$ for each k; then $A(t)$ has a HK-integral $Q[a,b]$ and $Q_k[a,b] \to Q[a,b]$ uniformly.*

Theorem 7. *Suppose $A(t)$ is Bochner integrable on $[a,b]$, then $A(t)$ has a HK-integral $Q[a,b]$ and:*

$$(B)\int_a^b A(t)dt = (HK)\int_a^b A(t)dt. \tag{1.4}$$

Proof. First, let E be a measurable subset of $[a,b]$ and assume that $A(t) = A\chi_E(t)$, where $\chi_E(t)$ is the characteristic function of E. In this case, we show that $Q[a,b] = Al(E)$, where $l(E)$ is the Lebesgue measure of E. Let $\varepsilon > 0$ be given and let D be a compact subset of E. Let $F \subset [a,b]$ be an open set containing E such that $l(F \backslash D) < \varepsilon / \|A\|$; and define $\delta : [a,b] \to (0,\infty)$ such that:

$$\delta(t) = \begin{cases} d(t, [a,b]\backslash F),\ t \in E \\ d(t, D),\ t \in [a,b]\backslash E, \end{cases}$$

where $d(x\,,\,y) = |x-y|$ is the distance function. Let $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ be a HK-partition for δ ; for $1 \leqslant i \leqslant n$, if $\tau_i \in E$ then $(t_{i-1}, t_i) \subset F$ so that

$$\left\| \sum\nolimits_{i=1}^n \Delta t_i A(\tau_i) - Al(F) \right\| = \|A\| \left[l(F) - \sum\nolimits_{\tau_i \in E} \Delta t_i \right]. \tag{1.5}$$

On the other hand, if $\tau_i \notin E$ then $(t_{i-1}, t_i) \cap D = \emptyset$ (empty set), and it follows that:

$$\left\| \sum\nolimits_{i=1}^{n} \Delta t_i A(\tau_i) - Al(D) \right\| = \|A\| \left[\sum\nolimits_{\tau_i \notin E} \Delta t_i - l(D) \right]. \tag{1.6}$$

Combining equations (1.5) and (1.6), we have that

$$\begin{aligned} &\left\| \sum\nolimits_{i=1}^{n} \Delta t_i A(\tau_i) - Al(E) \right\| = \|A\| \left[\sum\nolimits_{\tau_i \in E} \Delta t_i - l(E) \right] \\ &\leqslant \|A\| \left[l(F) - l(E) \right] \leqslant \|A\| \left[l(F) - l(D) \right] \leqslant \|A\| \, l(F \backslash D) < \varepsilon. \end{aligned}$$

Now suppose that $A(t) = \sum\nolimits_{k=1}^{\infty} A_k \chi_{E_k}(t)$. By definition, $A(t)$ is Bochner integrable if and only if $\|A(t)\|$ is Lebesgue integrable with:

$$(B)\int_a^b A(t)dt = \sum\nolimits_{k=1}^{\infty} A_k l(E_k),$$

and (cf. Hille and Phillips [HP])

$$(L)\int_a^b \|A(t)\| dt = \sum\nolimits_{k=1}^{\infty} \|A_k\| \, l(E_k).$$

As the partial sums converge uniformly by Theorem 7, $Q[a,b]$ exists and

$$Q[a,b] \equiv (HK)\int_a^b A(t)dt = (B)\int_a^b A(t)dt.$$

Now let $A(t)$ be an arbitrary Bochner integrable operator-valued function in $L(\mathcal{H})$, uniformly measurable and defined on $[a,b]$. By definition, there exists a sequence $\{A_k(t)\}$ of countably-valued operator-valued functions in $L(\mathcal{H})$ which converges to $A(t)$ in the uniform operator topology such that:

$$\lim_{k \to \infty} (L)\int_a^b \|A_k(t) - A(t)\| \, dt = 0,$$

and

$$(B)\int_a^b A(t)dt = \lim_{k\to\infty}(B)\int_a^b A_k(t)dt.$$

Since the $A_k(t)$ are countably-valued,

$$(KH)\int_a^b A_k(t)dt = (B)\int_a^b A_k(t)dt,$$

so

$$(B)\int_a^b A(t)dt = \lim_{k\to\infty}(HK)\int_a^b A_k(t)dt.$$

We are done if we show that $Q[a,b]$ exists. First, by the basic result of Henstock, every L-integral is a HK-integral, so that $f_k(t) = \|A_k(t) - A(t)\|$ has a HK-integral. The above means that $\lim\limits_{k\to\infty}(KH)\int_a^b f_k(t)dt = 0$. Let $\varepsilon > 0$ and choose m so large that

$$\left\|(B)\int_a^b A(t)dt - (HK)\int_a^b A_m(t)dt\right\| < \varepsilon/4$$

and

$$(HK)\int_a^b f_k(t)dt < \varepsilon/4,$$

Choose δ_1 so that, if $\{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ is a HK-partition for δ_1, then

$$\left\|(HK)\int_a^b A_m(t)dt - \sum\nolimits_{i=1}^n \Delta t_i A_m(\tau_i)\right\| < \varepsilon/4.$$

Now choose δ_2 so that, whenever $\{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ is a HK-partition for δ_2,

$$\left\|(HK)\int_a^b f_m(t)dt - \sum\nolimits_{i=1}^n \Delta t_i f_m(\tau_i)\right\| < \varepsilon/4.$$

Set $\delta = \delta_1 \wedge \delta_2$ so that, by Lemma 3, if $\{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$ is a HK-partition for δ, it is also one for δ_1 and δ_2, so that:

$$\left\| (B)\int_a^b A(t)dt - \sum_{i=1}^n \Delta t_i A(\tau_i) \right\| \leqslant \left\| (B)\int_a^b A(t)dt - (HK)\int_a^b A_m(t)dt \right\|$$
$$+ \left\| (HK)\int_a^b A_m(t)dt - \sum\nolimits_{i=1}^n \Delta t_i A_m(\tau_i) \right\| + \left| (HK)\int_a^b f_m(t)dt - \sum\nolimits_{i=1}^n \Delta t_i f_m(\tau_i) \right|$$
$$+ (HK)\int_a^b f_m(t)dt < \varepsilon.$$

□

1.2. **The KS-Hilbert Space.**

. Clearly, the most important factor preventing the wide spread use of the HK-integral in engineering, mathematics and physics has been the lack of a natural Banach space structure for this class of functions (as is the case for the Lebesgue integral). Our objective in this section is to construct a particular (separable) Hilbert space $\mathbb{KS}^2[\mathbf{R}^n]$. This space is of special interest, because it contains the class of HK-integrable functions, the space $\mathfrak{M}[\mathbf{R}^n]$ of measures on $\mathbf{R}^n$ and $\mathbf{L}^q[\mathbf{R}^n]$ for $1 \leqslant q \leqslant \infty$. Each of the above spaces is contained in $\mathbb{KS}^2[\mathbf{R}^n]$ as a continuous dense and compact embedding (e.g., weakly convergent sequences in each of the above spaces are strongly convergent in $\mathbb{KS}^2[\mathbf{R}^n]$). In addition, using results in other work [GBZS], we prove that both the Fourier transform and the convolution operator have bounded extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. This space is perfect for the highly oscillatory functions that occur in quantum theory and nonlinear analysis. In particular,

we will later show that $\mathbb{KS}^2[\mathbf{R}^n]$ allows us to (rigorously) construct the path integral for quantum mechanics in the manner first suggested by Feynman.

First, recall that the HK-integral is equivalent to the Denjoy integral (see Henstock [HS] or Pfeffer [PF]). In the one-dimensional case, Alexiewicz [AL] has shown that the class $D(\mathbf{R})$, of Denjoy integrable functions, can be normed in the following manner: for $f \in D(\mathbf{R})$, define $\|f\|_D$ by

$$\|f\|_D = \sup_s \left| \int_{-\infty}^s f(r)dr \right|.$$

It is clear that this is a norm, and it is known that $D(\mathbf{R})$ is not complete. Replacing $\mathbf{R}$ by $\mathbf{R}^n$, for $f \in D(\mathbf{R}^n)$, we introduce the following generalization:

$$\|f\|_D = \sup_{r>0} \left| \int_{\mathbf{B}_r} f(\mathbf{x})d\mathbf{x} \right| = \sup_{r>0} \left| \int_{\mathbb{R}^n} \mathcal{E}_{\mathbf{B}_r}(\mathbf{x}) f(\mathbf{x})d\mathbf{x} \right| < \infty, \tag{1.7}$$

where $\mathbf{B}_r$ is any closed ball in $\mathbf{R}^n$ and $\mathcal{E}_{\mathbf{B}_r}(\mathbf{x})$ is the characteristic function of $\mathbf{B}_r$. Now, fix n, and let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbf{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbf{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \cdots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed ball centered at $\mathbf{x}_i$ of radius $r_l = 2^{-l}, l \in \mathbb{N}$. Now choose an order so that the set $\{\mathbf{B}_k(\mathbf{x}_k),\ k \in \mathbb{N}\}$ contains all closed balls $\{\mathbf{B}_l(\mathbf{x}_i)\ \ |(l,i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k(\mathbf{x}_k)$, so that $\mathcal{E}_k(\mathbf{x})$ is in $\mathbf{L}^p[\mathbf{R}^n] \cap \mathbf{L}^\infty[\mathbf{R}^n]$ for $1 \leqslant p < \infty$. Define $F_k(\,\cdot\,)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$F_k(f) = \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})d\mathbf{x}. \tag{1.8}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $\mathbf{L}^p[\mathbf{R}^n]$ for each k, $\|F_k\|_\infty \leqslant 1$ and if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $\mathbf{L}^p[\mathbf{R}^n]$ for $1 \leqslant p \leqslant \infty$. Fix λ, set $t_\lambda^k = \lambda^{k-1}e^{-\lambda}/(k-1)!$ and define a measure $d\mathbf{P}_\lambda(\mathbf{x},\mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$ by:

$$d\mathbf{P}_\lambda(\mathbf{x},\mathbf{y}) = \left[\sum\nolimits_{k=1}^{\infty} t_\lambda^k \mathcal{E}_k(\mathbf{x})\mathcal{E}_k(\mathbf{y})\right] d\mathbf{x}d\mathbf{y}.$$

We can now define an inner product $(\,\cdot\,)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\mathbf{x})g(\mathbf{y})^* d\mathbf{P}_\lambda(\mathbf{x},\mathbf{y}) \\ &= \sum\nolimits_{k=1}^{\infty} t_\lambda^k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right]\left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y})g(\mathbf{y})d\mathbf{y}\right]^* .\end{aligned} \tag{1.9}$$

Our choice of t_λ^k is suggested by physical analysis in another context (see Gill and Zachary [GZ], and Section 6). We call the completion of $\mathbf{L}^1[\mathbf{R}^n]$, with the above inner product, the Kuelbs-Steadman space ($\mathbb{KS}^2[\mathbf{R}^n]$). Following suggestions of Gill and Zachary, Steadman (unpublished) constructed this space by adapting an approach developed by Kuelbs [KB] for other purposes. Her interest was in showing that $\mathbf{L}^1[\mathbf{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbf{R}^n]$, then:

$$\|f\|_{\mathbf{KS}}^2 = \sum\nolimits_{k=1}^{\infty} t_\lambda^k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \|f\|_D^2 ,$$

so $f \in \mathbb{KS}^2[\mathbf{R}^n]$.

Theorem 8. *For each p, $1 \leqslant p \leqslant \infty$, $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^p[\mathbf{R}^n]$ as a dense subspace.*

Proof. By construction, $\mathbb{KS}^2[\mathbf{R}^n]$ contains $\mathbf{L}^1[\mathbf{R}^n]$, so we need only show that $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ for $q \neq 1$. If $f \in \mathbf{L}^q[\mathbf{R}^n]$ and $q < \infty$, we have

$$\begin{aligned}
\|f\|_2^{\mathbf{KS}} &= \left[\sum\nolimits_{k=1}^{\infty} t_\lambda^k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{\frac{q2}{q}}\right]^{1/2} \\
&\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_\lambda^k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{2}{q}}\right]^{1/2} \\
&\leqslant \sup_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q .
\end{aligned}$$

Hence, $f \in \mathbb{KS}^2[\mathbf{R}^n]$. For $q = \infty$, we have

$$\begin{aligned}
\|f\|_2^{\mathbf{KS}} &= \left[\sum\nolimits_{k=1}^{\infty} t_\lambda^k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2\right]^{1/2} \\
&\leqslant \left[\left[\sum\nolimits_{k=1}^{\infty} t_\lambda^k [vol(\mathbf{B}_k)]^2\right] [ess\sup|f|]^2\right]^{1/2} \leqslant M\,\|f\|_\infty .
\end{aligned}$$

Thus $f \in \mathbb{KS}^2[\mathbf{R}^n]$, and $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$. □

The fact that $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$, while $\mathbb{KS}^2[\mathbf{R}^n]$ is separable makes it clear in a very forceful manner that whether a space is separable or not, depends on the topology. It is of particular interest to observe that $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^1[\mathbf{R}^n]^{**} = \mathfrak{M}[\mathbf{R}^n]$, the space of measures on $\mathbf{R}^n$ and that $\mathbb{KS}^2[\mathbf{R}^n]$ has a number of other interesting and useful features. However, before exploring these properties, we must discuss some recent results on the extension of linear operators on Banach spaces in the next section.

1.3. **Operator Theory.**

. In this section, we prove a number of results on operator extensions that will be of use later. One important application is to prove that the Fourier transform and convolution operators can be extended from $\mathbf{L}^2(\mathbf{R}^n)$ to $\mathbb{KS}^2(\mathbf{R}^n)$. We can then use these results to rigorously compute the free particle path integral introduced in the beginning, in the manner intended by Feynman. (Thus, $\mathbb{KS}^2(\mathbf{R}^n)$ allows positive solutions for the problems posed in our introduction.) Let $L[\mathcal{B}], L[\mathcal{H}]$ denote the bounded linear operators on a separable Banach or Hilbert space, $\mathcal{B}, \mathcal{H}$ respectively. By a duality map ϕ_x defined on $\mathcal{B}$, we mean any linear functional $f_x \in \left\{ f \in \mathcal{B}' \mid f(x) = \langle x, f \rangle = \|x\|^2, x \in \mathcal{B} \right\}$, where $\langle \cdot\ ,\ \cdot \rangle$ is the natural pairing between a Banach space and its dual. Let $\mathbf{J} : \mathcal{H} \to \mathcal{H}'$ be the standard conjugate isomorphism between a Hilbert space and its dual, so that $\langle x\ ,\ \mathbf{J}(x) \rangle = (x, x)_{\mathcal{H}} = \|x\|^2$. The following two theorems are by von Neumann [VN1] and Lax [LX], respectively. The first is well-known, and is proved in Yosida [YS]. The theorem by Lax is not as well-known, but important for later, so we provide a proof.

Theorem 9 (von Neumann)**.** *Let $\mathcal{H}$ be a separable Hilbert space and let A be a bounded linear operator on $\mathcal{H}$. Then A has a well defined adjoint A^* defined on $\mathcal{H}$ such that:*

1. *The operator $A^*A \geqslant 0$,*
2. $(A^*A)^* = A^*A$ *and*

(3) *$I + A^*A$ has a bounded inverse.*

Theorem 10 (Lax)**.** *Suppose $\mathcal{B}$ is a dense continuous embedding in a separable Hilbert space $\mathcal{H}$. Let $A \in L[\mathcal{B}]$. If A is selfadjoint on $\mathcal{H}$ (i.e., $(Ax, y)_{\mathcal{H}} = (x, Ay)_{\mathcal{H}}, \forall x,y \in \mathcal{B}$), then*

(1) *The operator A is bounded on $\mathcal{H}$ and $\|A\|_{\mathcal{H}} \leqslant k \|A\|_{\mathcal{B}}$, for some positive constant k.*

(2) *The spectrum of A over $\mathcal{H}$ and over $\mathcal{B}$, satisfies $\sigma_{\mathcal{H}}(A) \subset \sigma_{\mathcal{B}}(A)$.*

(3) *The point spectrum of A is unchanged by the extension (i.e., $\sigma^p_{\mathcal{H}}(A) = \sigma^p_{\mathcal{B}}(A)$).*

Proof. To prove (1), let $\varphi \in \mathcal{B}$ and, without loss, we can assume that $k = 1$ and $\|\varphi\|_{\mathcal{H}} = 1$. Since A is selfadjoint,

$$\|A\varphi\|^2_{\mathcal{H}} = (A\varphi, A\varphi) = \left(\varphi, A^2\varphi\right) \leqslant \|\varphi\|_{\mathcal{H}} \left\|A^2\varphi\right\|_{\mathcal{H}} = \left\|A^2\varphi\right\|_{\mathcal{H}}.$$

Thus, we have $\|A\varphi\|^4_{\mathcal{H}} \leqslant \left\|A^4\varphi\right\|_{\mathcal{H}}$, so it is easy to see that $\|A\varphi\|^{2n}_{\mathcal{H}} \leqslant \left\|A^{2n}\varphi\right\|_{\mathcal{H}}$ for all n. It follows that:

$$\|A\varphi\|_{\mathcal{H}} \leqslant (\left\|A^{2n}\varphi\right\|_{\mathcal{H}})^{1/2n} \leqslant (\left\|A^{2n}\varphi\right\|_{\mathcal{B}})^{1/2n}$$

$$\leqslant (\left\|A^{2n}\right\|_{\mathcal{B}})^{1/2n} (\|\varphi\|_{\mathcal{B}})^{1/2n} \leqslant \|A\|_{\mathcal{B}} (\|\varphi\|_{\mathcal{B}})^{1/2n}$$

Letting $n \to \infty$, we get that $\|A\varphi\|_{\mathcal{H}} \leqslant \|A\|_{\mathcal{B}}$ for φ in a dense set of the unit ball of $\mathcal{H}$. We are done, since the norm is attained on a dense set of the unit ball.

Let $\bar{A}$ be the extension of A to $\mathcal{H}$. To prove (2), first note that since A is self-adjoint on $\mathcal{H}$, any complex number (with nonzero imaginary part) is

in the resolvent set of A. If λ_0 is real and not in $\sigma_{\mathcal{B}}(A)$, let $R(\lambda_0, A) = (\lambda_0 I - A)^{-1}$. This operator is easily seen to be self-adjoint relative to the $\mathcal{H}$ inner product, so it is $\mathcal{H}$ norm bounded by (1). Thus, $R(\lambda_0, A)$ has a bounded extension to $\mathcal{H}$. Since $R(\lambda_0, A)(\lambda_0 I - A) = (\lambda_0 I - A)R(\lambda_0, A)$ is a bounded linear operator on $\mathcal{H}$, and equal to the identity on $\mathcal{B}$, it follows that λ_0 is not in $\sigma_{\mathcal{H}}(A)$.

To prove (3), let λ_0 be in $\sigma^p_{\mathcal{B}}(A)$, the point spectrum of A, so that $\lambda_0 I - A$ has a finite dimensional null space $\mathcal{N}$, with $\dim(\mathcal{N}) = \dim(\mathcal{B} \bmod \mathcal{J})$, where $\mathcal{J}$ is the range of $\lambda_0 I - A$ over $\mathcal{B}$. From the symmetry of A, we see that every element of $\mathcal{J}$ is orthogonal to $\mathcal{N}$. Since $\dim(\mathcal{N}) = \dim(\mathcal{B} \bmod \mathcal{J})$, we conclude that $\mathcal{J}$ contains precisely those elements in $\mathcal{B}$ that are orthogonal to $\mathcal{N}$. It follows that $\lambda_0 I - A$ is bijective when restricted to $\mathcal{J}$, so that the restriction of $\lambda_0 I - A$ to $\mathcal{J}$ has an inverse R that, by the closed graph theorem must be bounded. It now follows from (1) that R is bounded on $\mathcal{J}$ in the $\mathcal{H}$ norm and can be extended to a bounded linear operator on the closure of $\mathcal{J}$ in $\mathcal{H}$. It follows that the closure of $\mathcal{J}$ in $\mathcal{H}$ is orthogonal to $\mathcal{N}$, so that $\lambda_0 I - A$ has a bounded inverse on $\mathcal{N}^{\perp}$ with respect to $\mathcal{H}$. This means that λ_0 belongs to $\sigma^p_{\mathcal{H}}(A)$, the point spectrum of A over $\mathcal{H}$, and the null space of A over $\mathcal{H}$ is $\mathcal{N}$. $\square$

The following theorem shows that every separable Banach space may be rigged between two separable Hilbert spaces. The theorem is a restricted version of a result due to Gross and Kuelbs [GR], [KB]]. It is this rigging

that makes possible a number of new and interesting results on operator extensions.

Theorem 11. *(Gross-Kuelbs) Suppose $\mathcal{B}$ is a separable Banach space. Then there exist separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a positive trace class operator $\mathbf{T}_{12}$ defined on $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ (all as continuous dense embeddings), and $\mathbf{T}_{12}$ determines $\mathcal{H}_1$ when $\mathcal{B}$ and $\mathcal{H}_2$ are given.*

Proof. As $\mathcal{B}$ is separable, let $\{x_n\}$ be a dense set and let $\{f_n\}$, be any fixed set of corresponding duality mappings (i.e. $f_n \in \mathcal{B}'$ and $f_n(x_n) = \langle x_n, f_n \rangle = \|x_n\|_{\mathcal{B}}^2$). Let $\{t_n\}$ be a positive sequence of numbers such that $\sum_{n=1}^{\infty} t_n = 1$, and define $(x,y)_2$ by:

$$(x,y)_2 = \sum\nolimits_{n=1}^{\infty} t_n f_n(x) \bar{f}_n(y).$$

It is easy to see that $(x,y)_2$ is an inner product on $\mathcal{B}$. We let $\mathcal{H}_2$ be the Hilbert space generated by the completion of $\mathcal{B}$ with respect to this inner product. It is clear $\mathcal{B}$ is dense in $\mathcal{H}_2$, and as

$$\|x\|_2^2 = \sum\nolimits_{n=1}^{\infty} t_n \left| f_n(x) \right|^2 \leqslant \sup_n \left| f_n(x) \right|^2 = \|x\|_{\mathcal{B}}^2 \, ,$$

we see that the embedding is continuous.

. Now, let $\{\phi_n\} \subset \mathcal{B}$, be a complete orthonormal sequence for $\mathcal{H}_2$, and let $\{\lambda_n\}$ be a positive sequence such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $M = \sum_{n=1}^{\infty} \lambda_n^2 \left\| \phi_n \right\|_{\mathbf{B}}^2 < \infty$. Define the operator $\mathbf{T}_{12}$ on $\mathcal{B}$ by:

$$\mathbf{T}_{12} x = \sum\nolimits_{n=1}^{\infty} \lambda_n \left(x, \phi_n \right)_2 \phi_n .$$

Since

$$\mathcal{B} \subset \mathcal{H}_2 \Rightarrow \mathcal{H'}_2 \subset \mathcal{B}' \Rightarrow (\cdot, \phi_n)_2 \in \mathcal{B}', \ \forall n,$$

we have that $\mathbf{T}_{12}$ maps $\mathcal{B} \to \mathcal{B}$ and:

$$\|\mathbf{T}_{12}x\|_{\mathcal{B}}^2 \leqslant \left[\sum\nolimits_{n=1}^{\infty} \lambda_n^2 \, \|\phi_n\|_{\mathcal{B}}^2\right] \left[\sum\nolimits_{n=1}^{\infty} |(x, \phi_n)_2|^2\right] = M \, \|x\|_2^2 \leqslant M \, \|x\|_{\mathcal{B}}^2 .$$

Thus, $\mathbf{T}_{12}$ is a bounded operator on $\mathcal{B}$. Define $\mathcal{H}_1$ by:

$$\mathcal{H}_1 = \left\{ x \in \mathcal{B} \,\left|\, \sum\nolimits_{n=1}^{\infty} \lambda_n^{-1} \left|(x, \phi_n)_2\right|^2 < \infty \right.\right\}, \quad (x, y)_1 = \sum\nolimits_{n=1}^{\infty} \lambda_n^{-1} \, (x, \phi_n)_2 \, (\phi_n, y)_2 \, .$$

With the above inner product, $\mathcal{H}_1$ is a Hilbert space and since terms of the form $x_N = \sum_{k=1}^{N} \lambda_k^{-1} \, (x, \psi_k)_2 \, \phi_k : \ x \in \mathcal{B}$ are dense in $\mathcal{B}$, we see that $\mathcal{H}_1$ is dense in $\mathcal{B}$. It follows that $\mathcal{H}_1$ is also dense in $\mathcal{H}_2$. It is easy to see that $\mathbf{T}_{12}$ is a positive self adjoint operator with respect to the $\mathcal{H}_2$ inner product, so by the theorem of Lax, $\mathbf{T}_{12}$ has a bounded extension to $\mathcal{H}_2$ and $\|\mathbf{T}_{12}\|_2 \leqslant \|\mathbf{T}_{12}\|_{\mathcal{B}}$. Finally, it is easy to see that for $x, y \in \mathcal{H}_1$, $(x, y)_1 = (\mathbf{T}_{12}^{-1/2} x, \mathbf{T}_{12}^{-1/2} y)_2$ and $(x, y)_2 = (\mathbf{T}_{12}^{1/2} x, \mathbf{T}_{12}^{1/2} y)_1$. It follows that $\mathcal{H}_1$ is continuously embedded in $\mathcal{H}_2$, hence also in $\mathcal{B}$. $\square$

Define the Steadman duality map of $\mathcal{B}$ associated with $\mathcal{H}_2$ by: $f_x^s = \left(\|x\|_{\mathcal{B}}^2 \big/ \|x\|_2^2\right) \mathbf{J}(x)$. (It is easy to check that f_x^s is a duality map for $\mathcal{B}$.) A bounded linear operator A is said to be maximal accretive if $\langle Ax \ , \ f_x \rangle \geqslant 0$ for all $x \in \mathcal{B}$. The next result is a direct generalization of Theorem 9 (see Gill et al, [GBZS]).

Theorem 12 (von Neumann*). *Let $\mathcal{B}$ be a separable Banach space and let A be a bounded linear operator on $\mathcal{B}$. Then A has a well defined adjoint A^* defined on $\mathcal{B}$ such that:*

(1) *The operator $A^*A \geqslant 0$ (maximal accretive),*

(2) *$(A^*A)^* = A^*A$, and*

(3) *$I + A^*A$ has a bounded inverse.*

Proof. Assume A is bounded. If we let $\mathbf{J}_i : \mathcal{H}_i \to \mathcal{H}'_i$, then $A_1 \equiv A\left|_{\mathcal{H}_1}\right. : \mathcal{H}_1 \to \mathcal{H}_2$, and $A'_1 : \mathcal{H}'_2 \to \mathcal{H}'_1$. It follows that $A'_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}'_1$ and $\mathbf{J}_1^{-1}A'_1\mathbf{J}_2 : \mathbf{H}_2 \to \mathbf{H}_1 \subset \mathcal{B}$ so that, if we define $A^* = \left[\mathbf{J}_1^{-1}A'_1\mathbf{J}_2\right]|_{\mathcal{B}}$, then $A^* : \mathcal{B} \to \mathcal{B}$ (i.e., $A^* \in L[\mathcal{B}]$). To prove 1, first note that as $\mathbf{J}_i : \mathcal{H}_i \to \mathcal{H}'_i$, this implies that $\mathbf{J}'_i : (\mathcal{H}'_i)' \to \mathcal{H}'_i$, so that $\mathbf{J}'_i = \mathbf{J}_i$. Now, for $x \in \mathcal{H}_1$,

$$\langle A^*Ax, \mathbf{J}_2(x)\rangle = \left\langle Ax, (A^*)'\mathbf{J}_2(x)\right\rangle$$

so, by using the above definition of A^*, we get that

$$(A^*)'\mathbf{J}_2(x) = \left\{\left[\mathbf{J}_1^{-1}A'_1\mathbf{J}_2\right]|_{\mathcal{B}}\right\}'\mathbf{J}_2(x) = \left[\mathbf{J}_2A_1\mathbf{J}_1^{-1}\right]\mathbf{J}_2(x) = \mathbf{J}_2(A_1x).$$

Since $x \in \mathcal{H}_1 \Rightarrow A_1x = Ax$, and

$$\langle A^*Ax, f_x^s\rangle = \left(\|x\|_{\mathcal{B}}^2 \big/ \|x\|_2^2\right)\langle Ax, \mathbf{J}_2(A_1x)\rangle = \left(\|x\|_{\mathcal{B}}^2 \big/ \|x\|_2^2\right)\|Ax\|_2^2 \geqslant 0,$$

it follows that A^*A is accretive on a dense set, so that A^*A is accretive on $\mathcal{B}$. It is maximal accretive because it has no proper extension. To prove 2,

we have that, for $x \in \mathcal{H}_1$,

$$(A^*A)^*x = \left(\left\{\mathbf{J}_1^{-1}\left[\left\{\left[\mathbf{J}_1^{-1}A_1'\mathbf{J}_2\right]|_{\mathbf{B}}\ A\right\}_1\right]'\mathbf{J}_2\right\}|_{\mathcal{B}}\right)x$$

$$= \left(\left\{\mathbf{J}_1^{-1}\left[\left\{A_1'\left[\mathbf{J}_2A_1\mathbf{J}_1^{-1}\right]|_{\mathcal{B}}\right\}\right]\mathbf{J}_2\right\}|_{\mathcal{B}}\right)x = A^*Ax.$$

It follows that the same result holds on $\mathcal{B}$. Finally, the proof that $I + A^*A$ is invertible follows the same lines as in Yosida [YS]. □

To show that the above theorem extends to closed operators, requires a little more work. We begin with:

Theorem 13. *Suppose that $\mathcal{S}$ is a subset of $\mathcal{H}$ and $(\mathcal{S}, \langle\cdot,\cdot\rangle')$ is a Hilbert space. Then $\mathcal{S}$ is the range of a bounded linear operator in $\mathcal{H}$.*

Proof. Since $\mathcal{S}$ is a subset of $\mathcal{H}$, the inclusion map T from $(\mathcal{S}, \langle\cdot,\cdot\rangle')$ into $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is bounded. It follows that $T^* = J_{\mathcal{S}}^{-1}T'J_{\mathcal{H}}$ is bounded from $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ to $(\mathcal{S}, \langle\cdot,\cdot\rangle')$. If $T^* = U[TT^*]^{1/2}$ is the polar decomposition of T^*, then U is a partial isometry mapping $\mathcal{H}$ onto $\mathcal{S}$. Since T is nonnegative, so is U and $\langle U\varphi, U\psi\rangle' = \langle\varphi,\psi\rangle$ for all $\varphi,\psi \in \mathcal{H}$. □

Theorem 14. *If $A, B \in L(\mathcal{H})$, then*

$$R(A^*) + R(B^*) = R([A^*A + B^*B]^{1/2}).$$

Proof. Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ act on $\mathcal{H} \oplus \mathcal{H}$ in the normal way. We then have that $T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}$, so that $TT^* = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix}$. It follows

that:

$$[R(A^*)+R(B^*)]\oplus\{0\}=R(T)=R([TT^*]^{1/2})=R\begin{pmatrix}[A^*A+B^*B]^{1/2} & 0\\ 0 & 0\end{pmatrix}$$

$$=R([A^*A+B^*B]^{1/2})\oplus\{0\}\,.$$

□

Theorem 15. *Let C be a closed linear operator on $\mathcal{H}$. Then there exists a pair of bounded linear contraction operators $A,B\in L[\mathcal{H}]$ such that $C=AB^{-1}$, with B nonnegative. Furthermore, $D(C)=R(B)$, $R(C)=R(A)$ and $P=A^*A+B^*B$ is the orthogonal projection $B^{-1}B$ onto $\bar{R}(B^*)=R(A^*)+R(B^*)$.*

Proof. Let $\mathcal{S}=D(C)$ be the domain of C and endow it with the graph norm, so that $\langle\varphi,\psi\rangle'=\langle\varphi,\psi\rangle+\langle C\varphi,C\psi\rangle$. Since C is linear and closed, $(\mathcal{S},\langle\cdot,\cdot\rangle')$ is a Hilbert space and $\|\varphi\|_{\mathcal{H}}\leqslant\|\varphi\|_{\mathcal{S}}$. By Theorem 13, there is a bounded nonnegative contraction B with $B(\mathcal{H})=\mathcal{S}$ and, for $\varphi,\psi\in\mathcal{S}$, $\langle\varphi,\psi\rangle'=\left\langle B^{-1}\varphi,B^{-1}\psi\right\rangle$. Now let $A=CB$ so that, for $\varphi\in\mathcal{H}$, we have:

$$\langle A\varphi,A\varphi\rangle=\langle CB\varphi,CB\varphi\rangle\leqslant\langle B\varphi,B\varphi\rangle+\langle CB\varphi,CB\varphi\rangle$$

$$=\langle B\varphi,B\varphi\rangle'=\left\langle B^{-1}B\varphi,B^{-1}B\varphi\right\rangle=\langle P\varphi,P\varphi\rangle\leqslant\langle\varphi,\varphi\rangle\,.$$

Hence, $\|A\varphi\|^2\leqslant\|\varphi\|^2$, so that A is a contraction and $A=CB=(AB^{-1})B=A(B^{-1}B)=AP$. Also,

$$\langle\varphi,[A^*A+B^*B]\psi\rangle=\langle B\varphi,B\psi\rangle+\langle CB\varphi,CB\psi\rangle$$

$$=\langle B\varphi,B\psi\rangle'=\left\langle B^{-1}B\varphi,B^{-1}B\psi\right\rangle=\langle P\varphi,P\psi\rangle=\langle\varphi,P\psi\rangle\,.$$

Hence, $A^*A + B^*B = P$ and, since $R(A^*) + R(B^*) = R([A^*A + B^*B]^{1/2})$, $R(A^*) + R(B^*)$ is closed and equal to the closure of $R(B)$ (note that B is self-adjoint). $\square$

Let $\mathbf{V}(\mathcal{H})$ be the set of contractions and $\mathbf{C}(\mathcal{H})$ be the set of closed densely defined linear operators on $\mathcal{H}$. The following result is due to Kaufman [KF]

Theorem 16 (Kaufman)**.** *The equation* $\mathrm{K}(A) = A(I - A^*A)^{-1/2}$ *defines a continuous bijection from* $\mathbf{V}(\mathcal{H})$ *onto* $\mathbf{C}(\mathcal{H})$*, with inverse* $\mathrm{K}^{-1}(C) = C(I + C^*C)^{-1/2}$.

Proof. Let $A \in \mathbf{V}(\mathcal{H})$ and set $B = (I - A^*A)^{1/2}$, which is easily seen to be positive and in $\mathbf{V}(\mathcal{H})$. It follows that $\mathrm{K}(A) = AB^{-1}$ and $A^*A + B^2 = I$; so that, by the proof of Theorem 15, we see that $\mathrm{K}(A)$ is a closed linear operator on $\mathcal{H}$. Since the domain of $\mathrm{K}(A)$ is $B(\mathcal{H})$, which is dense in $\mathcal{H}$, $\mathrm{K}(A)$ is in $\mathbf{C}(\mathcal{H})$. On the other hand, if $C \in \mathbf{C}(\mathcal{H})$ then, by Theorem 15, there exists a pair of bounded linear contraction operators $A, B \in L[\mathcal{H}]$ such that $C = AB^{-1}$, with B positive with range $D(C)$ and $A^*A + B^2 = I$. Furthermore, for each nonzero φ, $\|\varphi\|_{\mathcal{H}}^2 - \|A\varphi\|_{\mathcal{H}}^2 = \|B\varphi\|_{\mathcal{H}}^2 > 0$; thus $A \in \mathbf{V}(\mathcal{H})$ and $\mathrm{K}(A) = C$. The graph of C is the set of all $\{(B\varphi, A\varphi), \varphi \in \mathcal{H}\}$, so that $\mathrm{C}^* = \{(\phi, \psi) \in \mathcal{H} \times \mathcal{H}\}$ such that $(\phi, A\varphi)_{\mathcal{H}} = (\psi, B\varphi)_{\mathcal{H}}$, or $(A^*\phi, \varphi)_{\mathcal{H}} = (B\psi, \varphi)_{\mathcal{H}}$ for all $\varphi \in \mathcal{H}$, so that $C^* = B^{-1}A^*$. It is clear that $I + C^*C$ is an invertible linear operator with bounded inverse and, for each

$\varphi \in \mathcal{H}$, we have that

$$\varphi = B^2\varphi + B^{-1}(I - B^2)B^{-1}B^2\varphi$$

$$= (I + B^{-1}A^*AB^{-1})B^2\varphi = (I + C^*C)B^2\varphi.$$

It follows that $(I + C^*C)^{-1} = B^2$ and therefore, $A = CB = C(I + C^*C)^{-1/2} = K^{-1}(C)$. $\square$

Theorem 17. *Every closed densely defined linear operator A on $\mathcal{B}$ extends to a closed densely defined linear operator $\bar{A}$ on $\mathcal{H}_2$, with $\rho(\bar{A}) = \rho(A)$ and $\sigma(\bar{A}) = \sigma(A)$.*

Proof. If $\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_2'$ is the standard conjugate isomorphism then, as $\mathcal{B}$ is strongly dense in $\mathcal{H}_2$, $\mathbf{J}_2[\mathcal{B}] \subset \mathcal{H}_2' \subset \mathcal{B}'$ is (strongly) dense in $\mathcal{H}_2'$. If A is any closed densely defined linear operator on $\mathcal{B}$ (with domain $D(A)$), then A' is closed on $\mathcal{B}'$ (the dual of $\mathcal{B}$). In addition, $A'\big|_{\mathcal{H}_2'}$ is closed and, for each $\varphi \in D(A)$, $\mathbf{J}_2(\varphi) \in \mathcal{H}_2'$, $\langle A\psi, \mathbf{J}_2(\varphi)\rangle$ is well defined for $\forall\psi \in D(A)$. Hence, $\mathbf{J}_2(\varphi) \in D(A')$ for all $\varphi \in D(A)$ and, since $\mathbf{J}_2(\mathcal{B})$ is strongly dense in $\mathcal{H}'_2$, this implies that $\mathbf{J}_2(D(A')) \subset D(A')$ is strongly dense in $\mathcal{H}'_2$, so that $D(A')\,|_{\mathcal{H}'_2}$ is strongly dense in $\mathcal{H}'_2$. Thus, as $\mathcal{H}_2$ is reflexive, $\bar{A} = \left[A'\big|_{\mathcal{H}_2'}\right]'$ is a closed densely defined operator on $\mathcal{H}_2$. To prove the second part, note that, if $\lambda I - \bar{A}$ has an inverse, then $\lambda I - A$ also has one, so $\rho(\bar{A}) \subset \rho(A)$ and $R(\lambda I - A) \subset R(\lambda I - \bar{A}) \subset \overline{R(\lambda I - A)}$ for any $\lambda \in \mathbf{C}$. For the other direction, assume that $\rho(A) \neq \emptyset$ so there is at least one $\lambda \in \rho(A)$. Then $(\lambda I - A)^{-1}$ is a continuous mapping from $R(\lambda I - A)$ onto

$D(A)$ and $R(\lambda I - A)$ is dense in $\mathcal{B}$. Let $\varphi \in D(\bar{A})$, so that $(\varphi, A\varphi) \in \bar{G}(A)$ by definition. Thus, there exists a sequence $\{\varphi_n\} \subset D(A)$ such that $\|\varphi - \varphi_n\|_G = \|\varphi - \varphi_n\|_{\mathcal{B}} + \|A\varphi - A\varphi_n\|_{\mathcal{B}} \to 0$ as $n \to \infty$. It follows that $(\lambda I - \bar{A})\varphi = \lim_{n\to\infty}(\lambda I - A)\varphi_n$. However, by the boundedness of $(\lambda I - A)^{-1}$ on $R(\lambda I - A)$, we have that, for some $\delta > 0$,

$$\left\|(\lambda I - \bar{A})\varphi\right\|_{\mathcal{B}} = \lim_{n\to\infty} \|(\lambda I - A)\varphi_n\|_{\mathcal{B}} \geqslant \lim_{n\to\infty} \delta \|\varphi_n\|_{\mathcal{B}} = \delta \|\varphi\|_{\mathcal{B}} .$$

It follows that $\lambda I - \bar{A}$ has a bounded inverse and, since $D(A) \subset D(\bar{A})$ implies that $R(\lambda I - A) \subset R(\lambda I - \bar{A})$, we see that $R(\lambda I - \bar{A})$ is dense in $\mathcal{B}$ so that $\lambda \in \rho(\bar{A})$ and hence $\rho(A) \subset \rho(\bar{A})$. It follows that $\rho(A) = \rho(\bar{A})$ and necessarily, $\sigma(A) = \sigma(\bar{A})$. □

Theorem 18 (Lax*)**.** *Suppose $\mathcal{B}$ is a dense and continuous embedding in a separable Hilbert space $\mathcal{H}_2$. Let $A \in L[\mathcal{B}]$, then:*

(1) *The operator A extends to $L[\mathcal{H}_2]$ and $\|A\|_2 \leqslant k \|A\|_{\mathcal{B}}$ for some fixed constant k.*

(2) *The spectrum and resolvent satisfies $\sigma_2(A) = \sigma_{\mathcal{B}}(A)$, $\rho_2(A) = \rho_{\mathcal{B}}(A)$.*

Proof. To prove (1), let A be any bounded linear operator on $\mathcal{B}$ and set $T = A^*A$. From the first part of Theorem 17, we see that T extends to a closed linear operator $(\bar{T})$ on $\mathcal{H}_2$. As $\bar{T}$ is selfadjoint on $\mathcal{H}_2$, by Theorem 10 $\bar{T}$ is bounded on $\mathcal{H}_2$ and

$$\|A^*A\|_2 = \|A\|_2^2 \leqslant \|A^*A\|_{\mathcal{B}} \leqslant k \|A\|_{\mathcal{B}}^2 ,$$

where $\mathrm{k} = \inf\left\{\mathrm{M} \,|\, \|A^*A\|_{\mathcal{B}} \leqslant \mathrm{M}\,\|A\|_{\mathcal{B}}^2\right\}$. The proof of (2) follows from the second part of Theorem 17. $\square$

Theorem 19. *Let $\mathcal{B}$ be a separable Banach space and let C be a closed densely defined linear operator on $\mathcal{B}$. Then there exists a closed densely defined linear operator C^* such that C^*C is maximal accretive, $(C^*C)^* = C^*C$ and $I + C^*C$ has a bounded inverse.*

Proof. If C is a closed densely defined linear operator on $\mathcal{B}$, let $\bar{C}$ be its extension to $\mathcal{H}_2$. By Theorem 16, $\bar{C} = \bar{A}(I - \bar{A}^*\bar{A})^{-1/2}$, where $\bar{A}$ is a linear contraction on $\mathcal{H}_2$ and $\bar{A} = \bar{C}(I + \bar{C}^*\bar{C})^{-1/2}$. Thus, every closed densely defined linear operator on $\mathcal{B}$ can be obtained as the restriction C of some $\bar{C}$ to $\mathcal{B}$. This means that every closed densely defined linear operator on $\mathcal{B}$ is of the form $C = \bar{A}(I - \bar{A}^*\bar{A})^{-1/2}|_{\mathcal{B}}$, so that each $\bar{A}$ is the extension of some linear contraction operator A on $\mathcal{B}$ to $\mathcal{H}_2$. Thus, on $\mathcal{B}$, $C = A(I - A^*A)^{-1/2}$ and, since A has an adjoint, C has one also $\left(C^* = (I - A^*A)^{-1/2}A^*\right)$. The properties of C^* now follow from those of A^*. $\square$

We now prove that $\mathfrak{F}$ and $\mathfrak{C}$, the Fourier transform and the convolution operator, respectively, defined on $\mathbf{L}^1[\mathbf{R}^n]$, have bounded extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. It should be noted that this theorem implies that both operators have bounded extensions to all $\mathbf{L}^p[\mathbf{R}^n]$ spaces, for $1 \leq p < \infty$. This is the first proof purely based on functional analysis, while the traditional proof is obtained via some rather deep methods of (advanced) real analysis.

Theorem 20. *We have that:*

(1) *Both $\mathfrak{F}$ and $\mathfrak{C}$ extend to bounded linear operators on $\mathbb{KS}^2[\mathbf{R}^n]$.*

(2) *If $f_n \xrightarrow{w} f$ (weakly) in $\mathbf{L}^p[\mathbf{R}^n], 1 \leqslant p \leqslant \infty$, then $f_n \xrightarrow{s} f$ in $\mathbb{KS}^2[\mathbf{R}^n]$ (the embedding of $\mathbf{L}^p[\mathbf{R}^n]$ in $\mathbb{KS}^2[\mathbf{R}^n]$ is compact).*

Proof. For the proof of (1); first use Theorem 18, to show that since $\mathfrak{F}$ is a bounded linear operator on $\mathbf{L}^1[\mathbf{R}^n]$, it extends to a bounded linear operator on $\mathbb{KS}^2[\mathbf{R}^n]$. For $\mathfrak{C}$, fix g in $\mathbf{L}^1[\mathbf{R}^n]$ and define $\mathfrak{C}_g$ on $\mathbf{L}^1[\mathbf{R}^n]$ by:

$$\mathfrak{C}_g(f)(\mathbf{y}) = \int g(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Since $\mathfrak{C}_g$ is bounded on $\mathbf{L}^1[\mathbf{R}^n]$, by Theorem 18 it extends to a bounded linear operator on $\mathbb{KS}^2[\mathbf{R}^n]$. Now use the fact that convolution is commutative to get that $\mathfrak{C}_f$ is a bounded linear operator on $\mathbf{L}^1[\mathbf{R}^n]$ for all $f \in \mathbb{KS}^2[\mathbf{R}^n]$. Another application of Theorem 18 completes (1). From the Gross-Keulbs Theorem, we know that $\mathbf{L}^p[\mathbf{R}^n]$ is a dense continuous embedding. To prove it is compact, let $f_n \xrightarrow{w} f$ in $\mathbf{L}^p[\mathbf{R}^n]$. Since $\mathcal{E}_k(\mathbf{x}) \in \mathbf{L}^q[\mathbf{R}^n],\ 1 \leqslant q \leqslant \infty$, it follows that for each k,

$$\left| \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left(f_n(\mathbf{x}) - f(\mathbf{x}) \right) d\mathbf{x} \right|^2 \to 0.$$

Thus $f_n \xrightarrow{s} f$ in $\mathbb{KS}^2[\mathbf{R}^n]$. □

Definition 21. *A sequence $\{\mu_k\} \subset \mathfrak{M}[\mathbf{R}^n]$ is said to converge weakly to μ, ($\mu_n \xrightarrow{w} \mu$), if for every bounded uniformly continuous function $h(\mathbf{x})$,*

$$\int_{\mathbf{R}^n} h(\mathbf{x}) d\mu_n \to \int_{\mathbf{R}^n} h(\mathbf{x}) d\mu.$$

Theorem 22. *If* $\mu_n \xrightarrow{w} \mu$ *in* $\mathfrak{M}[\mathbf{R}^n]$*, then* $\mu_n \xrightarrow{s} \mu$ *(strongly) in* $\mathbb{KS}^2[\mathbf{R}^n]$.

Proof. Since the characteristic function of a closed ball is a bounded uniformly continuous function, $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbf{R}^n]$ implies that

$$\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu_n \to \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu$$

for each k, so that $\lim_{n\to\infty} \|\mu_n - \mu\| = 0$. $\square$

1.4. **Semigroups of Operators.**

. In this section, we introduce some basic results from the theory of semigroups of operators, which will be used throughout the remainder of the paper. We restrict our development to a fixed Hilbert space $\mathcal{H}$, and assume when convenient, that $\mathcal{H} = \mathbb{KS}^2[\mathbf{R}^n]$.

Definition 23. *A family of linear operators* $\{S(t), 0 \leqslant t < \infty\}$ *(not necessarily bounded) defined on* $\mathcal{H}$ *is a semigroup if*

(1) $S(t+s)\varphi = S(t)S(s)\varphi$ *for* $\varphi \in D$*, the domain of the semigroup.*
(2) *The semigroup is said to be strongly continuous if* $\lim\limits_{\tau\to 0} S(t+\tau)\varphi = S(t)\varphi \ \ \forall\varphi \in D,\ t > 0.$
(3) *It is a* C_0*-semigroup if it is strongly continuous,* $S(0) = I$*, and* $\lim\limits_{t\to 0} S(t)\varphi = \varphi \ \ \forall\varphi \in \mathcal{H}.$
(4) $S(t)$ *is a* C_0*-contraction semigroup if* $\|S(t)\| \leqslant 1.$
(5) $S(t)$ *is a* C_0*-unitary group if* $S(t)S(t)^* = S(t)^*S(t) = I$*, and* $\|S(t)\| = 1.$

Definition 24. *A densely defined operator A is said to be m-dissipative if $\mathrm{Re}\left\langle A\varphi, f_{\varphi}\right\rangle \leqslant 0 \ \ \forall\varphi \in D(A)$, and $\mathrm{Ran}(I - A) = \mathcal{H}$ (range of $(I - A)$).*

Theorem 25 (see Goldstein [GS] or Pazy [PZ])**.** *Let $S(t)$ be a C_0-semigroup of contraction operators on $\mathcal{H}$. Then*

(1) *$A\varphi = \lim_{t\to 0}[S(t)\varphi - \varphi]/t$ exists for φ in a dense set, and $R(\lambda, A) = (\lambda I - A)^{-1}$ (the resolvent) exists for $\lambda > 0$ and $\|R(\lambda, A)\| \leqslant \lambda^{-1}$.*

(2) *The closed densely defined operator A generates a C_0-semigroup of contractions on $\mathcal{H}$, $\{S(t), 0 \leqslant t < \infty\}$, if and only if A is m-dissipative.*

(3) *If A is closed and densely defined with both A and A' dissipative then A is m-dissipative.*

If A is the generator of a strongly continuous semigroup $T(t) = \exp(tA)$ on $\mathcal{H}$, then the Yosida approximator for A is defined by $A_\lambda = \lambda AR(\lambda, A)$, where $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent of A. In general, A is closed and densely defined but unbounded. The Yosida approximator A_λ is bounded, converges strongly to A, and $T_\lambda(t) = \exp(tA_\lambda)$ converges strongly to $T(t) = \exp(tA)$. If A generates a contraction semigroup, then so does A_λ (see Pazy [PZ]). This result is very useful for applications. Unfortunately, for general semigroups, A may not have a bounded resolvent. Furthermore, it is very convenient to have a contractive approximator. As an application of the theory in the previous section, we will show that the Yosida approach can

be generalized in such a way as to give a contractive approximator for all strongly continuous semigroups of operators on $\mathcal{H}$. The theory was developed for semigroups of operators on Banach spaces, but is also new for Hilbert spaces. For any closed densely defined linear operator A on $\mathcal{H}$, let $T = -[A^*A]^{1/2}, \bar{T} = -[AA^*]^{1/2}$. Since $-T(-\bar{T})$ is maximal accretive, $T(\bar{T})$ generates a contraction semigroup. We can now write A as $A = UT$, where U is a partial isometry . Define A_λ by $A_\lambda = \lambda AR(\lambda, T)$. Note that $A_\lambda = \lambda UTR(\lambda, T) = \lambda^2 UR(\lambda, T) - \lambda U$ and, although A does not commute with $R(\lambda, T)$, we have $\lambda AR(\lambda, T) = \lambda R(\lambda, \bar{T})A$.

Theorem 26. *For every closed densely defined linear operator A on $\mathcal{H}$, we have that*

(1) A_λ *is a bounded linear operator and* $\lim_{\lambda\to\infty} A_\lambda x = Ax, \forall x \in D(A)$,
(2) $\exp[tA_\lambda]$ *is a bounded contraction for* $t > 0$*, and*
(3) *if A generates a strongly continuous semigroup* $T(t) = \exp[tA]$ *on* D *for* $t > 0$*,* $D(A) \subseteq D$*, then* $\lim_{\lambda\to\infty} \|\exp[tA_\lambda]x - \exp[tA]x\|_{\mathcal{H}} = 0 \quad \forall x \in D$.

Proof. : To prove 1, let $x \in D(A)$. Now use the fact that $\lim_{\lambda\to\infty} \lambda R(\lambda, \bar{T})x = x$ and $A_\lambda x = \lambda R(\lambda, \bar{T})Ax$. To prove 2, use $A_\lambda = \lambda^2 UR(\lambda, T) - \lambda U$, $\|\lambda R(\lambda, T)\|_{\mathcal{H}} = 1$, and $\|U\|_{\mathcal{H}} = 1$ to get that $\|\exp[t\lambda^2 UR(\lambda, T) - t\lambda U]\|_{\mathcal{H}} \leq \exp[-t\lambda\|U\|_{\mathcal{H}}] \exp[t\lambda\|U\|_{\mathcal{H}}\|\lambda R(\lambda, T)\|_{\mathcal{H}}] \leq 1$.

To prove 3, let $t > 0$ and $x \in D(A)$. Then

$$\begin{aligned}
\| \exp{[tA]}x - \exp{[tA_\lambda]}x\|_{\mathcal{H}} &= \|\int_0^t \frac{d}{ds}[e^{(t-s)A_\lambda}e^{sA}]xds\|_{\mathcal{H}} \\
&\leq \int_0^t \|[e^{(t-s)A_\lambda}(A - A_\lambda)e^{sA}x]\|_{\mathcal{H}} \\
&\leq \int_0^t \|[(A - A_\lambda)e^{sA}x]\|_{\mathcal{H}}ds.
\end{aligned}$$

Now use $\|[A_\lambda e^{sA}x]\|_{\mathcal{H}} = \|[\lambda R(\lambda, \bar{T})e^{sA}Ax]\|_{\mathcal{H}} \leq \|[e^{sA}Ax]\|_{\mathcal{H}}$ to get $\|[(A - A_\lambda)e^{sA}x]\|_{\mathcal{H}} \leq 2\|[e^{sA}Ax]\|_{\mathcal{H}}$. Now, since $\|[e^{sA}Ax]\|_{\mathcal{H}}$ is continuous, by the bounded convergence theorem we have $\lim_{\lambda\to\infty} \|\exp[tA]x - \exp[tA_\lambda]x\|_{\mathcal{H}} \leq \int_0^t \lim_{\lambda\to\infty} \|[(A - A_\lambda)e^{sA}x]\|_{\mathcal{H}}ds = 0$. □

Theorem 27. *Every C_0-semigroup of contractions and C_0-unitary group on $\mathbf{L}^2[\mathbf{R}^n]$, $\{S(t), 0 \leqslant t < \infty\}$, extends to a C_0-semigroup of contractions or C_0-unitary group on $\mathbb{KS}^2[\mathbf{R}^n]$.*

Proof. We prove the first result, the second is easy. From Theorem 18, $S(t)$ on $\mathbf{L}^2[\mathbf{R}^n]$ extends to a bounded linear operator $\bar{S}(t)$ on $\mathbb{KS}^2[\mathbf{R}^n]$. It is easy to see that $\bar{S}(t)$ is a semigroup. Let $\bar{A}$ be the extension of A, then the fact that $\sigma(\bar{A}) = \sigma(A)$ and $\rho(\bar{A}) = \rho(A)$ follows from Theorem 17. Since, in our case, $\rho(\bar{A}) = \rho(A) \supseteq (0, \infty)$, it follows that, for $\lambda > 0, Ran(\lambda I - \bar{A}) = \mathbb{KS}^2[\mathbf{R}^n]$. As $\bar{A}$ is densely defined and dissipative, it is m-dissipative, so that $\bar{A}$ generates a C_0-contraction semigroup on $\mathbb{KS}^2[\mathbf{R}^n]$. □

1.5. **Feynman Path Integral I.**

. The properties of $\mathbb{KS}^2[\mathbf{R}^n]$ derived earlier suggests that it may be a better replacement for $\mathbf{L}^2[\mathbf{R}^n]$ in the study of the path integral formulation of quantum theory developed by Feynman. Note that it is easy to prove that both the position and momentum operators have closed, densely defined extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. Furthermore, the extensions of $\mathfrak{F}$ and $\mathfrak{C}$ insure that all of the Schrödinger and Heisenberg theories have a faithful representation on $\mathbb{KS}^2[\mathbf{R}^n]$. These issues will be discussed more fully in another venue.

Since $\mathbb{KS}^2[\mathbf{R}^n]$ contains the space of measures, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $\mathbb{KS}^2[\mathbf{R}^n]$ topology. (For example, $[\sin(\lambda\cdot\mathbf{x})/(\lambda\cdot\mathbf{x})] \in \mathbb{KS}^2[\mathbf{R}^n]$ and converges strongly to $\delta(\mathbf{x})$.) Thus, the finitely additive set function defined on the Borel sets (Feynman kernel):

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B] = \int_B (2\pi i(t-s))^{-1/2}\exp\{i|\mathbf{x}-\mathbf{y}|^2\Big/2(t-s)\}d\mathbf{y}$$

is in $\mathbb{KS}^2[\mathbf{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]\|_{\mathrm{KS}} \leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]\|_{\mathfrak{M}} = \infty$ (the variation norm) and

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B] = \int_{\mathbf{R}^n}\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,\tau,d\mathbf{z}]\mathbb{K}_{\mathbf{f}}[\tau,\mathbf{z}\,;\,s,B],\ \ \text{(HK-integral)}.$$

Definition 28. *Let* $\mathbf{P}_n = \{t_0,\tau_1,t_1,\tau_2,\cdots,\tau_n,t_n\}$ *be a HK-partition of the interval* $[0,t]$ *for each* n, *with* $\lim_{n\to\infty}\Delta\mu_n = 0$ *(mesh). Set* $\Delta t_j = t_j -$

$t_{j-1}, \tau_0 = 0$ *and for* $\psi \in \mathbb{KS}^2[\mathbf{R}^n]$ *define*

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(0)] = e^{-\lambda t} \sum_{k=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbf{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j)\,;\, t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\begin{aligned} &\int_{\mathbf{R}^{[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)] \psi[\mathbf{x}(0)] \\ &= \lim_{\lambda \to \infty} \int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)] \psi[\mathbf{x}(0)] \end{aligned} \tag{1.10}$$

whenever the limit exists.

Our use of Borel summability in the definition will be clear after we develop our Feynman operator calculus. The next result is now elementary. A more general (sum over paths) result, that covers almost all application areas, will be proven in Section 6.

Theorem 29. *The function* $\psi(\mathbf{x}) \equiv 1 \in \mathbb{KS}^2[\mathbf{R}^n]$ *and*

$$\int_{\mathbf{R}^{[s,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)] = \mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, \mathbf{y}] = \tfrac{1}{\sqrt{2\pi i (t-s)}} \exp\{ i |\mathbf{x} - \mathbf{y}|^2 \big/ 2(t-s) \}.$$

The above result is what Feynman was trying to obtain, in this case.

2. Continuous Tensor Product Hilbert Space

In this section, we study the continuous tensor product Hilbert space of von Neumann, which contains a class of subspaces that we will use for our constructive representation of the Feynman operator calculus. Although von Neumann [VN2] did not develop his theory for our purpose, it will be clear that the theory is natural for our approach. Some might object that these

spaces are too big (non-separable) for physics. However, we observe that past objections to non-separable spaces do not apply to a theory which lays out all of space-time from past to present to future as required by Feynman. (It should be noted that the theory presented is formulated so that the basic space is separable at each instant of time, which is all that is required by quantum theory.) The theory developed in this section follows closely the original paper of von Neumann. However, we provide new proofs of some results and simplifed proofs of others.

Let $I = [a, b]$, $0 \le a < b \le \infty$ and, in order to avoid trivialities, we always assume that, in any product, all terms are nonzero.

Definition 30. *If $\{z_\nu\}$ is a sequence of complex numbers indexed by $\nu \in I$,*

(1) *We say that the product $\prod_{\nu \in I} z_\nu$, is convergent with limit z if, for every $\varepsilon > 0$, there is a finite set $J \subset I$ such that $\left|\prod_{\nu \in J} z_\nu - z\right| < \varepsilon$.*
(2) *We say that the product $\prod_{\nu \in I} z_\nu$ is quasi convergent if $\prod_{\nu \in I} |z_\nu|$ is convergent. (If the product is quasi convergent, but not convergent, we assign it the value zero.)*

Since I is not countable, we note that

$$0 < \left|\prod\nolimits_{\nu \in I} z_\nu\right| < \infty \;\; \Leftrightarrow \sum\nolimits_{\nu \in I} |1 - z_\nu| < \infty. \tag{2.1}$$

Thus, it follows that convergence implies that at most a countable number of the $z_\nu \neq 1$.

Let $\mathcal{H}_\nu = \mathcal{H}$, be a fixed Hilbert space, for each $\nu \in I$ and, for $\{\phi_\nu\} \in \prod_{\nu\in I} \mathcal{H}_\nu$, let Δ_I be those sequences $\{\phi_\nu\}$ such that $\sum_{\nu\in I} \left| \left\| \varphi_\nu \right\|_\nu - 1 \right| < \infty$. Define a functional on Δ_I by

$$\Phi(\psi) = \sum\nolimits_{k=1}^{n} \prod_{\nu\in I} \left\langle \varphi_\nu^k, \psi_\nu \right\rangle_\nu, \tag{2.2}$$

where $\psi = \{\psi_\nu\}, \{\varphi_\nu^k\} \in \Delta_I$, for $1 \le k \le n$. It is easy to see that this functional is linear in each component. Denote Φ by

$$\Phi = \sum\nolimits_{k=1}^{n} \otimes_{\nu\in I} \varphi_\nu^k.$$

Define the algebraic tensor product, $\otimes_{\nu\in I}\mathcal{H}_\nu$, by

$$\otimes_{\nu\in I} \mathcal{H}_\nu = \left\{ \sum\nolimits_{k=1}^{n} \otimes_{\nu\in I} \varphi_\nu^k \,\middle|\, \{\varphi_\nu^k\} \in \Delta_I, 1 \le k \le n, n \in \mathbb{N} \right\}. \tag{2.3}$$

We define a linear functional on $\otimes_{\nu\in I}\mathcal{H}_\nu$ by

$$\left(\sum\nolimits_{k=1}^{n} \otimes_{\nu\in I} \varphi_\nu^k, \sum\nolimits_{l=1}^{m} \otimes_{\nu\in I} \psi_\nu^l \right)_\otimes = \sum\nolimits_{l=1}^{m} \sum\nolimits_{k=1}^{n} \prod_{\nu\in I} \left\langle \varphi_\nu^k, \psi_\nu^l \right\rangle_\nu. \tag{2.4}$$

Lemma 31. *The functional $(\cdot,\cdot)_\otimes$ is a well-defined mapping on $\otimes_{\nu\in I}\mathcal{H}_\nu$.*

Proof. It suffices to show that, if $\Phi = 0$, then $(\Phi, \Psi)_\otimes = 0$. If $\Phi = \sum_{k=1}^{n} \otimes_{\nu\in I} \varphi_\nu^k$ and $\Psi = \sum_{l=1}^{m} \otimes_{\nu\in I} \psi_\nu^l$, then with $\psi_l = \{\psi_\nu^l\}$,

$$(\Phi, \Psi)_\otimes = \sum\nolimits_{l=1}^{m} \sum\nolimits_{k=1}^{n} \prod_{\nu\in I} \left\langle \varphi_\nu^k, \psi_\nu^l \right\rangle_\nu = \sum\nolimits_{l=1}^{m} \Phi(\psi_l) = 0. \tag{2.5}$$

□

Before continuing our discussion of the above functional, we first need to look a little more closely at the structure of the algebraic tensor product space, $\otimes_{\nu\in I}\mathcal{H}_\nu$.

Definition 32. *Let* $\phi = \underset{\nu\in I}{\otimes}\phi_\nu$ *and* $\psi = \underset{\nu\in I}{\otimes}\psi_\nu$ *be in* $\otimes_{\nu\in I}\mathcal{H}_\nu$.

(1) *We say that* ϕ *is strongly equivalent to* ψ *(*$\phi \equiv^s \psi$*), if and only if* $\sum\limits_{\nu\in I} \left|1 - \langle\phi_\nu, \psi_\nu\rangle_\nu\right| < \infty$.
(2) *We say that* ϕ *is weakly equivalent to* ψ *(*$\phi \equiv^w \psi$*), if and only if* $\sum\limits_{\nu\in I} \left|1 - \left|\langle\phi_\nu, \psi_\nu\rangle_\nu\right|\,\right| < \infty.$

Lemma 33. *We have* $\phi \equiv^w \psi$ *if and only if there exist* z_ν, $|\,z_\nu\,| = 1$, *such that* $\underset{\nu\in I}{\otimes} z_\nu\phi_\nu \equiv^s \underset{\nu\in I}{\otimes}\psi_\nu$.

Proof. Suppose that $\underset{\nu\in I}{\otimes} z_\nu\phi_\nu \equiv^s \underset{\nu\in I}{\otimes}\psi_\nu$. Then we have:

$$\sum_{\nu\in I}\left|1 - \left|\langle\phi_\nu, \psi_\nu\rangle_\nu\right|\right| = \sum_{\nu\in I}\left|1 - \left|\langle z_\nu\phi_\nu, \psi_\nu\rangle_\nu\right|\right| \leqslant \sum_{\nu\in I}\left|1 - \langle z_\nu\phi_\nu, \psi_\nu\rangle_\nu\right| < \infty.$$

If $\phi \equiv^w \psi$, set

$$z_\nu = \left|\langle\phi_\nu, \psi_\nu\rangle_\nu\right| / \langle\phi_\nu, \psi_\nu\rangle_\nu,$$

for $\langle\phi_\nu, \psi_\nu\rangle_\nu \neq 0$, and set $z_\nu = 1$ otherwise. It follows that

$$\sum_{\nu\in I}\left|1 - \langle z_\nu\phi_\nu, \psi_\nu\rangle_\nu\right| = \sum_{\nu\in I}\left|1 - \left|\langle\phi_\nu, \psi_\nu\rangle_\nu\right|\right| < \infty,$$

so that $\underset{\nu\in I}{\otimes} z_\nu\phi_\nu \equiv^s \underset{\nu\in I}{\otimes}\psi_\nu$. □

Theorem 34. *The relations defined above are equivalence relations on* $\otimes_{\nu\in I}\mathcal{H}_\nu$, *which decomposes* $\otimes_{\nu\in I}\mathcal{H}_\nu$ *into disjoint equivalence classes.*

Proof. Suppose $\underset{\nu\in I}{\otimes}\phi_\nu \equiv^s \underset{\nu\in I}{\otimes}\psi_\nu$. First note that the relation is clearly symmetric. Thus, we need only prove that it is reflexive and transitive. To prove that the first relation is reflexive, observe that $\left|1-\langle\psi_\nu,\phi_\nu\rangle_\nu\right| = \left|1-\overline{\langle\phi_\nu,\psi_\nu\rangle_\nu}\right| = \left|\overline{[1-\langle\phi_\nu,\psi_\nu\rangle_\nu]}\right| = \left|1-\langle\phi_\nu,\psi_\nu\rangle_\nu\right|$. To show that it is transitive, without loss, we can assume that $\left\|\psi_\nu\right\|_\nu = \left\|\phi_\nu\right\|_\nu = 1$. It is then easy to see that, if $\underset{\nu\in I}{\otimes}\phi_\nu \equiv^s \underset{\nu\in I}{\otimes}\psi_\nu$ and $\underset{\nu\in I}{\otimes}\psi_\nu \equiv^s \underset{\nu\in I}{\otimes}\rho_\nu$, then

$$1-\langle\phi_\nu,\rho_\nu\rangle_\nu = [1-\langle\phi_\nu,\psi_\nu\rangle_\nu] + [1-\langle\psi_\nu,\rho_\nu\rangle_\nu] + \langle\phi_\nu-\psi_\nu,\psi_\nu-\rho_\nu\rangle_\nu\,.$$

Now $\langle\phi_\nu-\psi_\nu,\phi_\nu-\psi_\nu\rangle_\nu = 2\left[1-\operatorname{Re}\langle\phi_\nu,\psi_\nu\rangle_\nu\right] \leqslant 2\left|1-\langle\phi_\nu,\psi_\nu\rangle_\nu\right|$, so that $\sum\limits_\nu \left\|\phi_\nu-\psi_\nu\right\|_\nu^2 < \infty$ and, by the same observation, $\sum\limits_\nu \left\|\psi_\nu-\rho_\nu\right\|_\nu^2 < \infty$. It now follows from Schwartz's inequality that $\sum\limits_\nu \left\|\phi_\nu-\psi_\nu\right\|_\nu \left\|\psi_\nu-\rho_\nu\right\|_\nu < \infty$. Thus we have that

$$\sum_{\nu\in I}\left|1-\langle\phi_\nu,\rho_\nu\rangle_\nu\right| \le \sum_{\nu\in I}\left|1-\langle\phi_\nu,\psi_\nu\rangle_\nu\right| + \sum_{\nu\in I}\left|1-\langle\psi_\nu,\rho_\nu\rangle_\nu\right|$$
$$+\sum_{\nu\in I}\left\|\phi_\nu-\psi_\nu\right\|_\nu \left\|\psi_\nu-\rho_\nu\right\|_\nu < \infty.$$

This proves the first case. The proof of the second case (weak equivalence) now follows from the above lemma. □

Theorem 35. *Let $\otimes_{\nu\in I}\varphi_\nu$ be in $\otimes_{\nu\in I}\mathcal{H}_\nu$. Then:*

(1) *The product $\prod_{\nu\in I}\left\|\varphi_\nu\right\|_\nu$ converges if and only if $\prod_{\nu\in I}\left\|\varphi_\nu\right\|_\nu^2$ converges.*

(2) *If $\prod_{\nu\in I}\left\|\varphi_\nu\right\|_\nu$ and $\prod_{\nu\in I}\left\|\psi_\nu\right\|_\nu$ converge, then $\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu$ is quasi-convergent.*

(3) *If* $\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu$ *is quasi-convergent then there exist complex numbers* $\{z_\nu\}$, $|z_\nu|=1$ *such that* $\prod_{\nu\in I}\langle z_\nu\varphi_\nu,\psi_\nu\rangle_\nu$ *converges.*

Proof. For the first case, convergence of either term implies that $\{\|\varphi_\nu\|_\nu\,,\ \nu\in I\}$ has a finite upper bound $M>0$. Hence

$$|1-\|\varphi_\nu\|_\nu|\leqslant|1+\|\varphi_\nu\|_\nu|\,|1-\|\varphi_\nu\|_\nu|=\left|1-\|\varphi_\nu\|_\nu^2\right|\leqslant(1+M)\,|1-\|\varphi_\nu\|_\nu|\,.$$

To prove (2), note that, if $J\subset I$ is any finite subset,

$$0\leqslant\left|\prod_{\nu\in J}\langle\varphi_\nu,\psi_\nu\rangle_\nu\right|\leqslant\prod_{\nu\in J}\|\varphi_\nu\|_\nu\prod_{\nu\in J}\|\psi_\nu\|_\nu<\infty.$$

Therefore, $0\leqslant\left|\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu\right|<\infty$ so that $\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu$ is quasi-convergent, and, if $0<\left|\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu\right|<\infty$, it is convergent. The proof of (3) now follows directly from the above lemma. □

Definition 36. *For* $\varphi=\underset{\nu\in I}{\otimes}\varphi_\nu\in\mathcal{H}_\otimes^2$, *we define* $\mathcal{H}_\otimes^2(\varphi)$ *to be the closed subspace generated by the span of all* $\psi\equiv^s\varphi$ *and we call it the strong partial tensor product space generated by the vector* φ.

Theorem 37. *For the partial tensor product spaces, we have the following:*

(1) *If* $\psi_\nu\neq\varphi_\nu$ *occurs for at most a finite number of* ν, *then* $\psi=\underset{\nu\in I}{\otimes}\psi_\nu\equiv^s\varphi=\underset{\nu\in I}{\otimes}\varphi_\nu$.

(2) *The space* $\mathcal{H}_\otimes^2(\varphi)$ *is the closure of the linear span of* $\psi=\underset{\nu\in I}{\otimes}\psi_\nu$ *such that* $\psi_\nu\neq\varphi_\nu$ *occurs for at most a finite number of* ν.

(3) *If* $\Phi=\otimes_{\nu\in I}\varphi_\nu$ *and* $\Psi=\otimes_{\nu\in I}\psi_\nu$ *are in different equivalence classes of* $\otimes_{\nu\in I}\mathcal{H}_\nu$, *then* $(\Phi,\Psi)_\otimes=\prod_{\nu\in I}\langle\varphi_\nu,\psi_\nu\rangle_\nu=0$.

(4) $\mathcal{H}_{\otimes}^{2}(\varphi)^{w} = \underset{\psi \equiv^{w} \phi}{\oplus} \left[\mathcal{H}_{\otimes}^{2}(\psi)^{s} \right].$

Proof. To prove (1), let J be the finite set of ν for which $\psi_\nu \neq \varphi_\nu$. Then

$$\sum_{\nu\in I} |1 - \langle \varphi_\nu, \psi_\nu \rangle_\nu| = \sum_{\nu\in J} |1 - \langle \varphi_\nu, \psi_\nu \rangle_\nu| + \sum_{\nu\in I\setminus J} |1 - \langle \varphi_\nu, \varphi_\nu \rangle_\nu| \leq c + \sum_{\nu\in I} \left|1 - \|\varphi_\nu\|_\nu^2\right| < \infty,$$

so that $\underset{\nu\in I}{\otimes} \psi_\nu \equiv \underset{\nu\in I}{\otimes} \varphi_\nu$

To prove (2), let $\mathcal{H}_{\otimes}^{2}(\varphi)^{\#}$ be the closure of the linear span of all $\psi = \otimes_{\nu\in I}\psi_\nu$ such that $\psi_\nu \neq \varphi_\nu$ occurs for at most a finite number of ν. There is no loss in assuming that $\|\varphi_\nu\|_\nu = 1$ for all $\nu \in I$. It is clear from (1), that $\mathcal{H}_{\otimes}^{2}(\varphi)^{\#} \subseteq \mathcal{H}_{\otimes}^{2}(\varphi)$. Thus, we are done if we can show that $\mathcal{H}_{\otimes}^{2}(\varphi)^{\#} \supseteq \mathcal{H}_{\otimes}^{2}(\varphi)$. For any vector $\psi = \otimes_{\nu\in I}\psi_\nu$ in $\mathcal{H}_{\otimes}^{2}(\varphi)$, $\varphi \equiv \psi$ so that $\sum_{\nu\in I} |1 - \langle \varphi_\nu, \psi_\nu \rangle_\nu| < \infty$. If $\|\psi\|_{\otimes}^{2} = 0$ then $\psi \in \mathcal{H}_{\otimes}^{2}(\varphi)^{\#}$, so we can assume that $\|\psi\|_{\otimes}^{2} \neq 0$. This implies that $\|\psi_\nu\|_\nu \neq 0$ for all $\nu \in I$ and $0 \neq \prod_{\nu\in I}(1/\|\psi_\nu\|_\nu) < \infty$; hence, by scaling if necessary, we may also assume that $\|\psi_\nu\|_\nu = 1$ for all $\nu \in I$. Let $0 < \varepsilon < 1$ be given, and choose δ so that $0 < \sqrt{2\delta e} < \varepsilon$ (e is the base for the natural log). Since $\sum_{\nu\in I} |1 - \langle \varphi_\nu, \psi_\nu \rangle_\nu| < \infty$, there is a finite set of distinct values $J = \{\nu_1, \cdots, \nu_n\}$ such that $\sum_{\nu\in I-J} |1 - \langle \varphi_\nu, \psi_\nu \rangle_\nu| < \delta$. Since, for any finite set of numbers $z_1, \cdots, z_n$, it is easy to see that $|\prod_{k=1}^{n} z_k - 1| = |\prod_{k=1}^{n} [1 + (z_k - 1)] - 1| \leq \left(\prod_{k=1}^{n} e^{|z_k - 1|} - 1\right)$, we have that

$$\left| \prod_{\nu\in I\setminus J} \langle \varphi_\nu, \psi_\nu \rangle_\nu - 1 \right| \leq (\exp\{ \sum_{\nu\in I\setminus J} |\langle \varphi_\nu, \psi_\nu \rangle_\nu - 1|\} - 1) \leq e^{\delta} - 1 \leq e\delta.$$

Now, define $\phi_\nu = \psi_\nu$ if $\nu \in J$, and $\phi_\nu = \varphi_\nu$ if $\nu \in I \backslash J$, and set $\phi_J = \otimes_{\nu \in I} \phi_\nu$ so that $\phi_J \in \mathcal{H}^2_\otimes(\varphi)^\#$ and

$$\|\psi - \phi_J\|^2_\otimes = 2 - 2\,\mathrm{Re}\left[\prod_{\nu \in J} \langle \varphi_\nu, \psi_\nu \rangle_\nu \cdot \prod_{\nu \in I-J} \langle \varphi_\nu, \psi_\nu \rangle_\nu\right]$$

$$= 2 - 2\,\mathrm{Re}\left[\prod_{\nu \in I} \|\psi_\nu\|^2_\nu \cdot \prod_{\nu \in I-J} \langle \varphi_\nu, \psi_\nu \rangle_\nu\right] = 2\,\mathrm{Re}\left[1 - \prod_{\nu \in I-J} \langle \varphi_\nu, \psi_\nu \rangle_\nu\right] \leqslant 2e\delta < \varepsilon^2.$$

Since ε is arbitrary, ψ is in the closure of $\mathcal{H}^2_\otimes(\varphi)^\#$, so $\mathcal{H}^2_\otimes(\varphi)^\# = \mathcal{H}^2_\otimes(\varphi)$.

To prove (3), first note that, if $\prod_{\nu \in I} \|\varphi_\nu\|_\nu$ and $\prod_{\nu \in I} \|\psi_\nu\|_\nu$ converge, then, for any finite subset $J \subset I$, $0 \leq \left|\prod_{\nu \in J} \langle \varphi_\nu, \psi_\nu \rangle_\nu\right| \leq \prod_{\nu \in J} \|\varphi_\nu\|_\nu \prod_{\nu \in J} \|\psi_\nu\|_\nu < \infty$. Therefore, $0 \leq \left|\prod_{\nu \in I} \langle \varphi_\nu, \psi_\nu \rangle_\nu\right| = \left|(\Phi, \Psi)_\otimes\right| < \infty$ so that $\prod_{\nu \in I} \langle \varphi_\nu, \psi_\nu \rangle_\nu$ is convergent or zero. If $0 < \left|(\Phi, \Psi)_\otimes\right| < \infty$, then $\sum\limits_{\nu \in I} \left|1 - \langle \phi_\nu, \psi_\nu \rangle_\nu\right| < \infty$ and, by definition, Φ and Ψ are in the same equivalence class, so we must have $\left|(\Phi, \Psi)_\otimes\right| = 0$. The proof of (4) follows from the definition of weakly equivalent spaces. $\square$

Theorem 38. $(\Phi, \Psi)_\otimes$ *is a conjugate bilinear positive definite functional.*

Proof. The first part is trivial. To prove that it is positive definite, let $\Phi = \sum_{k=1}^n \otimes_{\nu \in I} \varphi^k_\nu$, and assume that the vectors $\otimes_{\nu \in I} \varphi^k_\nu, 1 \leq k \leq n$, are in distinct equivalence classes. This means that, with $\Phi_k = \otimes_{\nu \in I} \varphi^k_\nu$, we have

$$(\Phi, \Phi)_\otimes = \left(\sum\nolimits_{k=1}^n \Phi_k, \sum\nolimits_{k=1}^n \Phi_k\right)_\otimes = \sum\nolimits_{k=1}^n \sum\nolimits_{j=1}^n (\Phi_k, \Phi_j)_\otimes = \sum\nolimits_{k=1}^n (\Phi_k, \Phi_k)_\otimes.$$

Note that, from Theorem 37 (3), $k \neq j \Rightarrow (\Phi_k, \Phi_j)_\otimes = 0$. Thus, it suffices to assume that $\otimes_{\nu \in I} \varphi^k_\nu$, $1 \leq k \leq n$, are all in the same equivalence class. In

this case, we have that

$$(\Phi, \Phi)_{\otimes} = \sum\nolimits_{k=1}^{n} \sum\nolimits_{j=1}^{n} \prod_{\nu \in I} \left\langle \varphi_{\nu}^{k}, \varphi_{\nu}^{j} \right\rangle_{\nu},$$

where each product is convergent. It follows that the above will be positive definite if we can show that, for all possible finite sets $J = \{\nu_1, \nu_2 \cdots, \nu_m\}, m \in \mathbb{N}$,

$$\sum\nolimits_{k=1}^{n} \sum\nolimits_{j=1}^{n} \prod_{\nu \in J} \left\langle \varphi_{\nu}^{k}, \varphi_{\nu}^{j} \right\rangle_{\nu} \geq 0.$$

This is equivalent to showing that the above defines a positive definite functional on $\otimes_{\nu \in J} \mathcal{H}_{\nu}$, which follows from the standard result for finite tensor products of Hilbert spaces (see Reed and Simon, [RS]). □

Definition 39. *We define $\mathcal{H}_{\otimes}^{2} = \hat{\otimes}_{\nu \in I} \mathcal{H}_{\nu}$ to be the completion of the linear space $\otimes_{\nu \in I} \mathcal{H}_{\nu}$, relative to the inner product $(\cdot, \cdot)_{\otimes}$.*

2.1. **Orthonormal Basis for $\mathcal{H}_{\otimes}^{2}(\varphi)$.**

. We now construct an orthonormal basis for each $\mathcal{H}_{\otimes}^{2}(\varphi)$. Let $\mathbf{N}$ be the natural numbers, and let $\{e_{n}^{\nu},\ n \in \mathbb{N} = \mathbf{N} \cup \{0\}\}$ be a complete orthonormal basis for $\mathcal{H}_{\nu}$. Let e_{0}^{ν} be a fixed unit vector in $\mathcal{H}_{\nu}$ and set $E = \otimes_{\nu \in I} e_{0}^{\nu}$. Let $\mathbf{F}$ be the set of all functions $f : \ I \to \mathbb{N}$ such that $f(\nu) = 0$ for all but a finite number of ν. Let $F(f)$ be the image of $f \in \mathbf{F}$ (e.g., $F(f) = \{f(\nu),\ \nu \in I\}$), and set $E_{F(f)} = \otimes_{\nu \in I} e_{\nu, f(\nu)}$, where $f(\nu) = 0 \Rightarrow e_{\nu,0} = e_{0}^{\nu}$ and $f(\nu) = n \Rightarrow e_{\nu,n} = e_{n}^{\nu}$.

Theorem 40. *The set* $\{E_{F(f)},\ f \in \mathbf{F}\}$ *is a complete orthonormal basis for* $\mathcal{H}_{\otimes}^{2}(E)$*.*

Proof. First, note that $E \in \{E_{F(f)},\ f \in \mathbf{F}\}$ and each $E_{F(f)}$ is a unit vector. Also, we have $E_{F(f)} \equiv^{s} E$ and $\left\langle E_{F(f)}, E_{F(g)} \right\rangle = \prod_{\nu \in I} \left\langle e_{\nu, f(\nu)}, e_{\nu, g(\nu)} \right\rangle = 0$ unless $f(\nu) = g(\nu)$ for all ν. Hence, the family $\{E_{F(f)},\ f \in \mathbf{F}\}$ is an orthonormal set of vectors in $\mathcal{H}_{\otimes}^{2}(E)$. Let $\mathcal{H}_{\otimes}^{2}(E)^{\#}$ be the completion of the linear span of this set of vectors. Clearly $\mathcal{H}_{\otimes}^{2}(E)^{\#} \subseteq \mathcal{H}_{\otimes}^{2}(E)$ so we only need prove that every vector in $\mathcal{H}_{\otimes}^{2}(E) \subset \mathcal{H}_{\otimes}^{2}(E)^{\#}$. By Theorem 37 (2), it suffices to prove that $\mathcal{H}_{\otimes}^{2}(E)^{\#}$ contains the closure of the set of all $\varphi = \otimes_{\nu \in I} \varphi_{\nu}$ such that $\varphi_{\nu} \neq e_{0}^{\nu}$ occurs for only a finite number of ν. Let $\varphi = \otimes_{\nu \in I} \varphi_{\nu}$ be any such vector, and let $J = \{\nu_1, \cdots, \nu_k\}$ be the finite set of distinct values of ν for which $\varphi_{\nu} \neq e_{0}^{\nu}$ occurs. Since $\{e_{n}^{\nu},\ n \in \mathbb{N}\}$ is a basis for $\mathcal{H}_{\nu}$, for each ν_i there exist constants $a_{\nu_i, n}$ such that $\sum_{n \in \mathbb{N}} a_{\nu_i, n} e_{n}^{\nu_i} = \varphi_{\nu_i}$ for $1 \leq i \leq k$. Let $\varepsilon > 0$ be given. Then, for each ν_i there exists a finite subset $\mathbb{N}_i \subset \mathbb{N}$ such that $\left\| \varphi_{\nu_i} - \sum_{n \in \mathbb{N}_i} a_{\nu_i, n} e_{n}^{\nu_i} \right\|_{\otimes} < \frac{1}{n}(\varepsilon / \left\| \varphi \right\|_{\otimes})$. Let $\vec{\mathbb{N}} = (\mathbb{N}_1, \cdots \mathbb{N}_k)$ and set $\varphi_{\nu_i}^{\mathbb{N}_i} = \sum_{n \in \mathbb{N}_i} a_{\nu_i, n} e_{n}^{\nu_i}$ so that $\varphi^{\vec{\mathbb{N}}} = \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i}^{\mathbb{N}_i} \otimes (\underset{\nu \in I \setminus J}{\otimes} e_{0}^{\nu})$ and $\varphi = \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i} \otimes (\underset{\nu \in I \setminus J}{\otimes} e_{0}^{\nu})$. It follows that:

$$
\begin{aligned}
\left\| \varphi - \varphi^{\vec{\mathbb{N}}} \right\|_{\otimes} &= \left\| \left[\underset{\nu_i \in J}{\otimes} \varphi_{\nu_i} - \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i}^{\mathbb{N}_i} \right] \otimes (\underset{\nu \in I \setminus J}{\otimes} e_{0}^{\nu}) \right\|_{\otimes} \\
&= \left\| \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i} - \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i}^{\mathbb{N}_i} \right\|_{\otimes}.
\end{aligned}
$$

We can rewrite this as:

$$\left\| \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i} - \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i}^{\mathbb{N}_i} \right\|_{\otimes} = \| \varphi_{\nu_1} \otimes \varphi_{\nu_2} \cdots \otimes \varphi_{\nu_k} - \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2} \cdots \otimes \varphi_{\nu_k}$$

$$+ \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2} \cdots \otimes \varphi_{\nu_k} - \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2}^{\mathbb{N}_2} \cdots \otimes \varphi_{\nu_k}$$

$$\vdots$$

$$+ \ \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2}^{\mathbb{N}_2} \cdots \otimes \varphi_{\nu_{k-1}}^{\mathbb{N}_{k-1}} \otimes \varphi_{\nu_k} - \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2}^{\mathbb{N}_2} \cdots \otimes \varphi_{\nu_k}^{\mathbb{N}_k} \Big\|_{\otimes}$$

$$\leq \sum\nolimits_{i=1}^{n} \left\| \varphi_{\nu_i} - \varphi_{\nu_i}^{\mathbb{N}_i} \right\|_{\otimes} \|\varphi\|_{\otimes} \leq \varepsilon.$$

Now, as the tensor product is multilinear and continuous in any finite number of variables, we have:

$$\varphi^{\vec{\mathbb{N}}} = \underset{\nu_i \in J}{\otimes} \varphi_{\nu_i}^{\mathbb{N}_i} \otimes (\underset{\nu \in I \setminus J}{\otimes} e_0^{\nu}) = \varphi_{\nu_1}^{\mathbb{N}_1} \otimes \varphi_{\nu_2}^{\mathbb{N}_2} \cdots \otimes \varphi_{\nu_k}^{\mathbb{N}_k} \otimes (\underset{\nu \in I \setminus J}{\otimes} e_0^{\nu})$$

$$= \left[\sum\nolimits_{n_1 \in \mathbb{N}_1} a_{\nu_1, n_1} e_{n_1}^{\nu_1} \right] \otimes \left[\sum\nolimits_{n_2 \in \mathbb{N}_2} a_{\nu_2, n_2} e_{n_2}^{\nu_2} \right] \cdots \otimes \left[\sum\nolimits_{n_k \in \mathbb{N}_k} a_{\nu_k, n_k} e_{n_k}^{\nu_k} \right] \otimes (\underset{\nu \in I \setminus J}{\otimes} e_0^{\nu})$$

$$= \sum\nolimits_{\gamma_1 \in N_1 \cdots \gamma_n \in N_n} a_{\nu_1, n_1} a_{\nu_2, n_2} \cdots a_{\nu_k, n_k} \left[e_{n_1}^{\nu_1} \otimes e_{n_2}^{\nu_2} \cdots \otimes e_{n_k}^{\nu_k} \otimes (\underset{\nu \in I \setminus J}{\otimes} e_0^{\nu}) \right].$$

It is now clear that, by definition of $\mathbf{F}$, for each fixed set of indices $n_1, \ n_2, \cdots \ n_k$ there exists a function $f : \ I \to \mathbb{N}$ such that $f(\nu_i) = n_i$ for $\nu_i \in J$ and $f(\nu) = 0$ for $\nu \in I \setminus J$. Since each $\mathbb{N}_i$ is finite, $\vec{\mathbb{N}} = (\mathbb{N}_1, \cdots \mathbb{N}_k)$ is also finite, so that only a finite number of functions are needed. It follows that $\varphi^{\vec{\mathbb{N}}}$ is in $\mathcal{H}_{\otimes}^2(E)^{\#}$, so that φ is a limit point and $\mathcal{H}_{\otimes}^2(E)^{\#} = \mathcal{H}_{\otimes}^2(E)$. $\square$

2.2. **Tensor Product Semigroups.**

. Let $S_i(t), \ i = 1, 2,$ be C_0-contraction semigroups with generators A_i defined on $\mathcal{H}$, so that $\|S_i(t)\|_{\mathcal{H}} \leqslant 1$. Define operators $\mathbf{S}_1(t) = S_1(t) \hat{\otimes} \mathbf{I}_2$,

$\mathbf{S}_2(t) = \mathbf{I}_1 \hat{\otimes} S_2(t)$ and $\mathbf{S}(t) = S_1(t) \hat{\otimes} S_2(t)$ on $\mathcal{H} \hat{\otimes} \mathcal{H}$. The proof of the next result is easy.

Theorem 41. *The operators* $\mathbf{S}(t),\ \mathbf{S}_i(t)\ , i = 1, 2,$ *are* C_0*-contraction semigroups with generators* $\mathcal{A} = \overline{A_1 \hat{\otimes} \mathbf{I}_2 + \mathbf{I}_1 \hat{\otimes} A_2}$*,* $\mathcal{A}_1 = A_1 \hat{\otimes} \mathbf{I}_2$*,* $\mathcal{A}_2 = \mathbf{I}_1 \hat{\otimes} A_2$*, and* $\mathbf{S}(t) = \mathbf{S}_1(t)\mathbf{S}_2(t) = \mathbf{S}_2(t)\mathbf{S}_1(t)$*.*

Let $S_i(t),\ 1 \leqslant i \leqslant n$, be a family of C_0-contraction semigroups with generators A_i defined on $\mathcal{H}$.

Corollary 42. $\mathbf{S}(t) = \hat{\otimes}_{i=1}^{n} S_i(t)$ *is a* C_0*-contraction semigroup on* $\hat{\otimes}_{i=1}^{n} \mathcal{H}$ *and the closure of* $A_1 \hat{\otimes} \mathbf{I}_2 \hat{\otimes} \cdots \hat{\otimes} \mathbf{I}_n + \mathbf{I}_1 \hat{\otimes} A_2 \hat{\otimes} \cdots \hat{\otimes} \mathbf{I}_n + \cdots \mathbf{I}_1 \hat{\otimes} \mathbf{I}_2 \hat{\otimes} \cdots \hat{\otimes} A_n$ *is the generator* $\mathcal{A}$ *of* $\mathbf{S}(t)$*.*

3. Time-Ordered Operators

For the remainder of the paper, our index set $I = [a, b]$, is a subset of the reals, $\mathbf{R}$ and we replace $\mathcal{H}_{\otimes}^2 = \hat{\otimes}_{\nu \in I} \mathcal{H}_{\nu}$ by $\hat{\otimes}_{t \in I} \mathcal{H}(t)$. Let $L(\mathcal{H}_{\otimes}^2)$ be the set of bounded operators on $\mathcal{H}_{\otimes}^2$, and define $L(\mathcal{H}(t)) \subset L(\mathcal{H}_{\otimes}^2)$ by:

$$(3.1) L(\mathcal{H}(t)) = \left\{ \mathcal{A}(t) = \hat{\otimes}_{b \geqslant s > t} \mathrm{I}_s \otimes A(t) \otimes (\otimes_{t > s \geqslant -a} \mathrm{I}_s), \forall A(t) \in L(\mathcal{H}) \right\},$$

where $\mathrm{I_s}$ is the identity operator. Let $L^{\#}(\mathcal{H}_{\otimes}^2)$ be the uniform closure of the algebra generated by $\{L(\mathcal{H}(t)),\ t \in I\}$. If the family $\{A(t), t \in I\}$ is in $L(\mathcal{H})$, then the operators $\{\mathcal{A}(t),\ t \in I\} \in L^{\#}(\mathcal{H}_{\otimes}^2)$ commute when acting at different times: $t \neq \tau \Rightarrow$

$$\mathcal{A}(t)\mathcal{A}(\tau) = \mathcal{A}(\tau)\mathcal{A}(t).$$

Let $\mathbf{P}_\varphi$ denote the projection from $\mathcal{H}_\otimes^2$ onto $\mathcal{H}_\otimes^2(\varphi)$.

Theorem 43. *If* $\mathbf{T} \in L^{\#}[\mathcal{H}_\otimes^2]$, *then* $\mathbf{P}_\varphi \mathbf{T} = \mathbf{T}\mathbf{P}_\varphi$.

Proof. Since vectors of the form $\Phi = \sum_{i=1}^{L} \otimes_{s \in I} \varphi_s^i$, with $\varphi_s^i = \varphi_s$ for all but a finite number of s, are dense in $\mathcal{H}_\otimes^2(\varphi)$; it suffices to show that $\mathbf{T} \in L^{\#}[\mathcal{H}_\otimes^2] \Rightarrow \mathbf{T}\Phi \in \mathcal{H}_\otimes^2(\varphi)$. Now, $\mathbf{T} \in L^{\#}[\mathcal{H}_\otimes^2]$ implies that there exists a sequence of operators $\mathbf{T}_n$ such that $\|\mathbf{T} - \mathbf{T}_n\|_\otimes \to 0$ as $n \to \infty$, where each $\mathbf{T}_n$ is of the form: $\mathbf{T}_n = \sum_{k=1}^{N_n} a_k^n T_k^n$, with a_k^n a scalar, $N_n < \infty$, and each $T_k^n = \hat{\otimes}_{s \in J_k} T_{ks}^n \hat{\otimes}_{s \in I \setminus J_k} I_s$ for some finite set of s-values J_k. Hence,

$$\mathbf{T}_n \Phi = \sum\nolimits_{i=1}^{L} \sum\nolimits_{k=1}^{N_n} a_k^n \otimes_{s \in J_k} T_{ks}^n \varphi_s^i \otimes_{s \in I \setminus J_k} \varphi_s^i.$$

It is easy to see that, for each i, $\otimes_{s \in J_k} T_{ks}^n \varphi_s^i \otimes_{s \in I \setminus J_k} \varphi_s^i \equiv \otimes_{s \in I} \varphi_s$. It follows that $\mathbf{T}_n \Phi \in \mathcal{H}_\otimes^2(\varphi)$ for each n, so that $\mathbf{T}_n \in L[\mathcal{H}_\otimes^2(\varphi)]$. As $L[\mathcal{H}_\otimes^2(\varphi)]$ is a norm closed algebra, $\mathbf{T} \in L[\mathcal{H}_\otimes^2(\varphi)]$ and it follows that $\mathbf{P}_\varphi \mathbf{T} = \mathbf{T}\mathbf{P}_\varphi$. $\square$

Definition 44. *We call* $L^{\#}(\mathcal{H}_\otimes^2)$ *the time-ordered von Neumann algebra over* $\mathcal{H}_\otimes^2$.

The following theorem is due to von Neumann [VN2].

Theorem 45. *The mapping* $\mathbf{T}_\theta^t : L(\mathcal{H}) \to L(\mathcal{H}(t))$ *is an isometric isomorphism of algebras. (We call* $\mathbf{T}_\theta^t$ *the time-ordering morphism.)*

3.1. **Exchange Operator.**

Definition 46. *An exchange operator* $\mathbf{E}[t,t']$ *is a linear map defined for pairs* t,t' *such that:*

1. $\mathbf{E}[t,t'] \,:\, L[\mathcal{H}(t)] \to L[\mathcal{H}(t')]$, *(Isometric isomorphism),*
2. $\mathbf{E}[t,s]\mathbf{E}[s,t'] = \mathbf{E}[t,t']$,
3. $\mathbf{E}[t,t']\mathbf{E}[t',t] = I$,
4. *For* $s \neq t,\ t'$, $\mathbf{E}[t,t']\mathcal{A}(s) = \mathcal{A}(s),\ \forall \mathcal{A}(s) \in L[\mathcal{H}(s)]$.

The exchange operator acts to exchange the time positions of a pair of operators in a more complicated expression.

Theorem 47. *(Existence) There exists an exchange operator for* $L^{\#}[\mathcal{H}^2_{\otimes}]$.

Proof. Define a map $C[t,t'] \,:\, \mathcal{H}^2_{\otimes} \to \mathcal{H}^2_{\otimes}$ (comparison operator) by its action on elementary vectors:

$$C[t,t'] \otimes_{s\in I} \phi_s = \otimes_{a\leqslant s<t'}\phi_s \otimes \phi_t \otimes (\otimes_{t'<s<t}\phi_s) \otimes \phi_{t'} \otimes (\otimes_{t<s\leqslant b}\phi_s),$$

for all $\phi = \otimes_{s\in I}\phi_s \in \mathcal{H}^2_{\otimes}$. Clearly, $C[t,t']$ extends to an isometric isomorphism of $\mathcal{H}^2_{\otimes}$. For $\mathbf{U} \in L^{\#}[\mathcal{H}^2_{\otimes}]$, we define $\mathbf{E}[t,t']\,\mathbf{U} = C[t,t']\,\mathbf{U}C[t',t]$. It is easy to check that $\mathbf{E}[\,\cdot\,,\cdot\,]$ satisfies all the requirements for an exchange operator. $\square$

3.2. **The Film.**

. In the world view suggested by Feynman, physical reality is laid out as a three-dimensional motion picture in which we become aware of the future

as more and more of the film comes into view. (The way the world appears to us in our consciousness.)

In order to motivate our approach, let $\left\{e^i \mid i \in \mathbb{N}\right\}$ be a complete orthonormal basis for $\mathcal{H}$, and, for each $t \in I$ and $i \in \mathbb{N}$, let $e_t^i = e^i$ and set $E^i = \otimes_{t \in I} e_t^i$. Now notice that the Hilbert space $\widehat{\mathcal{H}}$ generated by the family of vectors $\{E^i,\ i \in \mathbb{N}\}$ is isometrically isomorphic to $\mathcal{H}$. For later use, it should be noted that any vector in $\mathcal{H}$ of the form $\varphi = \sum_{k=1}^{\infty} a_k e^k$ has the corresponding representation in $\widehat{\mathcal{H}}$ as $\hat{\varphi} = \sum_{k=1}^{\infty} a_k E^k$. The problem with using $\widehat{\mathcal{H}}$ to define our operator calculus is that this space is not invariant for any reasonable class of operators. We now construct a particular structure, which is our mathematical version of this film.

Definition 48. *A film, $\mathcal{FD}_{\otimes}^2$, is the smallest subspace containing $\widehat{\mathcal{H}}$, which is invariant for $L^{\#}[\mathcal{H}_{\otimes}^2]$. We call $\mathcal{FD}_{\otimes}^2$ the Feynman Dyson space (FD-space) over $\mathcal{H}$.*

In order to construct our space, let $\mathcal{FD}_2^i = \mathcal{H}_{\otimes}^2(E^i)$ be the strong partial tensor product space generated by the vector E^{i}. It is clear that $\mathcal{FD}_2^i$ is the smallest space in $\mathcal{H}_{\otimes}^2$ which contains the vector E^{i}. We now set $\mathcal{FD}_{\otimes}^2 = \bigoplus_{i=1}^{\infty} \mathcal{FD}_2^i$. It is clear that the space $\mathcal{FD}_{\otimes}^2$, is a nonseparable Hilbert (space) bundle over $I = [a, b]$. However, by construction, it is not hard to see that the fiber at each time-slice is isomorphic to $\mathcal{H}$ almost everywhere.

In order to facilitate the proofs in the next section, we need an explicit basis for each $\mathcal{FD}_2^i$. As in Section 2.1, let $\mathbf{F}$ be the set of all functions

$f(\,\cdot\,)\colon I \to \mathbb{N} \cup \{0\}$ such that $f(t)$ is zero for all but a finite number of t, and let $F(f)$ denote the image of the function $f(\cdot)$. Set $E^i_{F(f)} = \otimes_{t \in I} e^i_{t,f(\mathrm{t})}$ with $e^i_{t,0} = e^i$, and $f(t) = k \Rightarrow e^i_{t,k} = e^k$.

Lemma 49. *The set* $\{E^i_{F(f)} \,|F(f) \in \mathbf{F}\}$ *is a (c.o.b) for each* $\mathcal{FD}^i_2$.

If $\Phi^i = \sum_{F(f)\in\mathbf{F}} a_F(f)^i E^i_{F(f)}$, $\Psi^i = \sum_{F(f)\in\mathbf{F}} b^i_{F(f)} E^i_{F(f)} \in \mathcal{FD}^i_2$, set $a^i_{F(f)} = \left\langle \Phi^{\mathrm{i}}, E^i_{F(f)} \right\rangle$ and $b^{\mathrm{i}}_{F(f)} = \left\langle \Psi^{\mathrm{i}}, E^i_{F(f)} \right\rangle$, so that

$$\left\langle \Phi^{\mathrm{i}}, \Psi^{\mathrm{i}} \right\rangle = \sum_{F(f),F(g)\in\mathbf{F}} a^{\mathrm{i}}_{F(f)} \bar{b}^{\mathrm{i}}_{F(g)} \left\langle E^i_{F(f)}, E^i_{F(g)} \right\rangle, \text{ and } \left\langle \Phi^{\mathrm{i}}, \Psi^{\mathrm{i}} \right\rangle = \sum_{F(f)\in\mathbf{F}} a^{\mathrm{i}}_{f(t)} \bar{b}^{\mathrm{i}}_{f(t)}.$$

(Note that $\left\langle E^i_{F(f)}, E^i_{F(g)} \right\rangle = \prod_{t\in I} \left\langle e^{\mathrm{i}}_{t,f(t)}, e^{\mathrm{i}}_{t,g(t))} \right\rangle = 0$ unless $f(t) = g(t)\ \forall t \in I$.)

The following notation will be used at various points of this section so we record the meanings here for reference. (The t value referred to is in our fixed interval I.)

(1) (e.o.v): "except for at most one t value";

(2) (e.f.n.v): "except for an at most finite number of t values"; and

(3) (a.s.c): "almost surely and the exceptional set is at most countable".

3.3. **Time-Ordered Integrals and Generation Theorems.**

. In this section, we assume that $I = [a,b] \subseteq [0,\infty)$ and, for each $t \in I$, $A(t)$ generates a C_0-semigroup on $\mathcal{H}$.

To partially see the advantage of developing our theory on $\mathcal{FD}_{\otimes}^2$, suppose that $A(t)$ generates a C_0-semigroup for $t \in I$ and define $\mathbf{S}_t(\tau)$ by:

$$\mathbf{S}_t(\tau) = \hat{\otimes}_{s\in[b,t)}\mathrm{I}_s \otimes (\exp\{\tau A(t)\}) \otimes \left(\otimes_{s\in(t,a]}\mathrm{I}_s\right). \tag{3.2}$$

We briefly investigate the relationship between $S_t(\tau) = \exp\{\tau A(t)\}$ and $\mathbf{S}_t(\tau) = \exp\{\tau\mathcal{A}(t)\}$. By Theorems 13, 26 and 32, we know that $\mathbf{S}_t(\tau)$ is a C_0-semigroup for $t \in I$ if and only if $S_t(\tau)$ is one also. For additional insight, we need a dense core for the family $\{\mathcal{A}(t)\,|t\, \in I\}$, so let $\bar{D} = \underset{t\in I}{\otimes} D(A(t))$ and set $\mathrm{D}_0 = \bar{D} \cap \mathcal{FD}_{\otimes}^2$. Since $\bar{D}$ is dense in $\mathcal{H}_{\otimes}^2$, it follows that D_0 is dense in $\mathcal{FD}_{\otimes}^2$. Using our basis, if $\Phi, \Psi \in \mathrm{D}_0$, $\Phi = \sum_{\mathrm{i}} \sum_{F(f)} a_{F(f)}^{\mathrm{i}} E_{F(\mathrm{f})}^{\mathrm{i}}$, $\Psi = \sum_{\mathrm{i}} \sum_{F(g)} b_{F(g)}^{\mathrm{i}} E_{F(g)}^{\mathrm{i}}$; then, as $\exp\{\tau\mathcal{A}(t)\}$ is invariant on $\mathcal{FD}^i$, we have

$$\langle \exp\{\tau\mathcal{A}(t)\}\Phi, \Psi\rangle = \sum\nolimits_{\mathrm{i}} \sum\nolimits_{F(f)} \sum\nolimits_{F(g)} a_{F(f)}^{\mathrm{i}} \bar{b}_{F(g)}^{\mathrm{i}} \left\langle \exp\{\tau\mathcal{A}(t)\}E_{F(f)}^{\mathrm{i}}, E_{F(g)}^{\mathrm{i}}\right\rangle,$$

and

$$\left\langle \exp\{\tau\mathcal{A}(t)\}E_{F(f)}^{\mathrm{i}}, E_{F(g)}^{\mathrm{i}}\right\rangle = \prod_{s\neq t}\left\langle e_{s,f(s)}^{\mathrm{i}}, e_{s,g(s)}^{\mathrm{i}}\right\rangle \left\langle \exp\{\tau A(t)\}e_{t,f(t)}^{\mathrm{i}}, e_{t,g(t)}^{\mathrm{i}}\right\rangle$$

$$= \left\langle \exp\{\tau A(t)\}e_{t,f(t)}^{\mathrm{i}}, e_{t,f(t)}^{\mathrm{i}}\right\rangle \text{ (e.o.v)},$$

$$= \langle \exp\{\tau A(t)\}e^{\mathrm{i}}, e^{\mathrm{i}}\rangle \text{ (e.f.n.v.)},$$

$$\Rightarrow \langle \exp\{\tau\mathcal{A}(t)\}\Phi, \Psi\rangle = \sum\nolimits_{\mathrm{i}} \sum\nolimits_{F(f)} a_{F(f)}^{\mathrm{i}} \bar{b}_{F(f)}^{\mathrm{i}} \langle \exp\{\tau A(t)\}e^{\mathrm{i}}, e^{\mathrm{i}}\rangle (a.s).$$

Thus, by working on $\mathcal{FD}_{\otimes}^{2}$, we obtain a simple direct relationship between the conventional and time-ordered version of a semigroup. This suggests that a parallel theory of semigroups of operators on $\mathcal{FD}_{\otimes}^{2}$ might make it possible for physical theories to be formulated in the intuitive and conceptually simpler time-ordered framework, offering substantial gain compared to the conventional mathematical structure. Note that this approach would also obviate the need for the problematic process of disentanglement suggested by Feynman in order to relate the operator calculus to conventional mathematics. Let $\mathcal{A}_z(t) = z\mathcal{A}(t)\mathbf{R}(z, \mathcal{A}(t))$ (respectively $\mathcal{A}_z(t) = z\mathcal{A}(t)\mathbf{R}(z, \mathbf{T}(t)))$, where $\mathbf{R}(z, \mathcal{A}(t))$ (respectively $\mathbf{R}(z, \mathbf{T}(t))$ $z > 0$, is the resolvent of $\mathcal{A}(t)$ (respectively $\mathbf{T}(t)$). In the latter case, $\mathbf{T}(t)) = -[\mathcal{A}^*(t)\mathcal{A}(t)]^{1/2}$ and $\bar{\mathbf{T}}(t) = -[\mathcal{A}(t)\mathcal{A}^*(t)]^{1/2}$. Set $\bar{\mathcal{A}}_z(t) = z\mathcal{A}(t)\mathbf{R}(z, \bar{\mathbf{T}}(t))$.

By Theorem 27, in either case, $A_z(t)$ generates a uniformly bounded semigroup and $\lim\limits_{z\to\infty} A_z(t)\phi = A(t)\phi$ for $\phi \in \mathrm{D}(A(\mathrm{t}))$.

Theorem 50. *The operator $\mathcal{A}_z(t)$ satisfies*

(1) $\mathcal{A}(t)\mathcal{A}_z(t)\Phi = \bar{\mathcal{A}}_z(t)\mathcal{A}(t)\Phi$, $\Phi \in D$, *$\mathcal{A}_z(t)$ generates a uniformly bounded contraction semigroup on $\mathcal{FD}_{\otimes}^{2}$ for each t, and* $\lim\limits_{z\to\infty} \mathcal{A}_z(t)\Phi = \mathcal{A}(t)\Phi$, $\Phi \in D$.

(2) *For each n, each set $\tau_1, \cdots, \tau_n \in I$ and each set $a_1, \cdots, a_n$, $a_i \geqslant 0$; $\sum_{i=1}^{n} a_i\mathcal{A}(\tau_i)$ generates a C_0-semigroup on $\mathcal{FD}_{\otimes}^{2}$.*

Proof. The proof of (1) follows from Theorem 27 and the relationship between $\mathcal{A}(t)$ and $A(t)$. It is an easy computation to check that (2), follows from Theorem 42 and Corollary 43, with $\mathbf{S}(t) = \prod_{i=1}^{n} \mathbf{S}_{\tau_i}(a_i t)$. □

We now assume that $A(t),\ t \in I$, is weakly continuous and that $D(A(t)) \supseteq D$, where D is dense in $\mathcal{H}$ and independent of t. It follows that this family has a weak KH-integral $Q[a,b] = \int_a^b A(t)dt \in C(\mathcal{H})$ (the closed densely defined linear operators on $\mathcal{H}$). Furthermore, it is not difficult to see that $A_z(t),\ t \in I$, is also weakly continuous and hence the family $\{A_z(t) \mid t \in I\} \subset L(\mathcal{H})$ has a weak KH-integral $Q_z[a,b] = \int_a^b A_z(t)dt \in L(\mathcal{H})$. Let $\mathrm{P_n}$ be a sequence of KH-partitions for $\delta_n(t) : [a,b] \to (0,\infty)$ with $\delta_{n+1}(t) \geqslant \delta_n(t)$ and $\lim_{n\to\infty} \delta_n(t) = 0$, so that the mesh $\mu_n = \mu(\mathrm{P_n}) \to 0$ as $\mathrm{n} \to \infty$. Set $Q_{z,n} = \sum_{l=1}^{n} A_z(\bar{t}_l)\Delta t_l$, $Q_{z,m} = \sum_{q=1}^{m} A_z(\bar{s}_q)\Delta s_q$; $\mathbf{Q}_{z,n} = \sum_{l=1}^{\mathrm{n}} \mathcal{A}_z(\bar{t}_l)\Delta t_l$, $\mathbf{Q}_{z,m} = \sum_{q=1}^{\mathrm{m}} \mathcal{A}_z(\bar{s}_q)\Delta s_q$; and $\Delta Q_z = Q_{z,n} - Q_{z,m}$, $\Delta\mathbf{Q}_z = \mathbf{Q}_{z,n} - \mathbf{Q}_{z,m}$. Let $\Phi, \Psi \in \mathrm{D}_0$; $\Phi = \sum_i^J \Phi^{\mathrm{i}} = \sum_i^J \sum_{F(f)}^K a^{\mathrm{i}}_{F(f)} E^{\mathrm{i}}_{F(f)}$, $\Psi = \sum_i^L \Psi^{\mathrm{i}} = \sum_i^L \sum_{F(g)}^M b^{\mathrm{i}}_{F(g)} E^{\mathrm{i}}_{F(g)}$. Then we have:

Theorem 51. *(Fundamental Theorem for Time-Ordered Integrals)*

(1) *The family $\{\mathcal{A}_z(t) \mid t \in I\}$ has a weak KH-integral and*

$$\langle \Delta\mathbf{Q}_z\Phi, \Psi\rangle = \sum_i^J \sum_{F(f)}^K a^i_{F(f)} \bar{b}^i_{F(f)} \left\langle \Delta Q_z e^i, e^i \right\rangle \ \ (a.s.c). \tag{3.3}$$

(2) *If, in addition, for each i*

$$\sum_{k,}^{n} \Delta t_k \left\| A_z(s_k)e^i - \left\langle A_z(s_k)e^i, e^i\right\rangle e^i \right\|^2 \leqslant M\mu_n^{\delta-1}, \tag{3.4}$$

where M is a constant, μ_n is the mesh of P_n, and $0 < \delta < 1$, then the family $\{\mathcal{A}_z(t) \mid t \in I\}$ has a strong integral, $\mathbf{Q}_z[t,a] = \int_a^t \mathcal{A}_z(s)ds$.

(3) *The linear operator $\mathbf{Q}_z[t,a]$ generates a uniformly continuous C_0-contraction semigroup.*

Remark 52. *In general, the family $\{A_z(t) \mid t \in I\}$ need not have a Bochner or Pettis integral. (However, if it has a Bochner integral, our condition 3.4 is automatically satisfied.)*

Proof. To prove (1), note that

$$\langle \Delta\mathbf{Q}_z\Phi, \Psi\rangle = \sum\nolimits_i \sum\nolimits_{F(f)} \sum\nolimits_{F(g)} a^{\mathrm{i}}_{F(f)} \bar{b}^{\mathrm{i}}_{F(g)} \left\langle \Delta\mathbf{Q}_z E^{\mathrm{i}}_{F(f)}, E^{\mathrm{i}}_{F(g)} \right\rangle$$

(we omit the upper limit). Now

$$\begin{aligned}
&\left\langle \Delta\mathbf{Q}_z E^{\mathrm{i}}_{F(f)}, E^{\mathrm{i}}_{F(g)} \right\rangle = \sum_{l=1}^{\mathrm{n}} \Delta t_l \prod_{\mathrm{t}\neq\bar{t}_l} \left\langle e^{\mathrm{i}}_{t,f(t)}, e^{\mathrm{i}}_{t,g(t)} \right\rangle \left\langle A_z(\bar{t}_l) e^{\mathrm{i}}_{\bar{t}_l,f(\bar{t}_l)}, e^{\mathrm{i}}_{\bar{t}_l,g(\bar{t}_l)} \right\rangle \\
&- \sum_{q=1}^{\mathrm{m}} \Delta s_q \prod_{\mathrm{t}\neq\bar{s}_q} \left\langle e^{\mathrm{i}}_{t,f(t)}, e^{\mathrm{i}}_{t,g(t)} \right\rangle \left\langle A_z(\bar{s}_q) e^{\mathrm{i}}_{\bar{s}_q,f(\bar{s}_q)}, e^{\mathrm{i}}_{\bar{s}_q,g(\bar{s}_q)} \right\rangle = \sum_{l=1}^{\mathrm{n}} \Delta t_l \left\langle A_z(\bar{t}_l) e^{\mathrm{i}}_{\bar{t}_l,f(\bar{t}_l)}, e^{\mathrm{i}}_{\bar{t}_l,f(\bar{t}_l)} \right\rangle \\
&- \sum_{q=1}^{\mathrm{m}} \Delta s_q \left\langle A_z(\bar{s}_q) e^{\mathrm{i}}_{\bar{s}_q,f(\bar{s}_q)}, e^{\mathrm{i}}_{\bar{s}_q,f(\bar{s}_q)} \right\rangle = \left\langle \Delta Q_z e^i, e^i \right\rangle \ \text{(e.f.n.v)}.
\end{aligned}$$

This gives (3.4) and shows that the family $\{\mathcal{A}_z(\mathrm{t}) \mid t \in I\}$ has a weak KH-integral if and only if the family $\{A_z(\mathrm{t}) \mid t \in I\}$ has one.

To see that condition (3.4) makes $\mathbf{Q}_z$ a strong limit, let $\Phi \in \mathrm{D}_0$. Then

$$\langle \mathbf{Q}_{z,n}\Phi, \mathbf{Q}_{z,n}\Phi\rangle = \sum\nolimits_{\mathrm{i}}^{J} \sum\nolimits_{F(f),F(g)}^{K} a^{\mathrm{i}}_{F(f)} \bar{a}^{\mathrm{i}}_{F(g)} \left(\sum_{\mathrm{k,m}}^{\mathrm{n}} \sum\nolimits_{k=1}^{n} \Delta \mathrm{t}_k \Delta \mathrm{t}_m \left\langle \mathcal{A}_z(s_k) E^{\mathrm{i}}_{F(f)}, \mathcal{A}_z(s_m) E^{\mathrm{i}}_{F(g)} \right\rangle \right)$$

$$= \sum\nolimits_{\mathrm{i}}^{J} \sum\nolimits_{F(f)}^{K} \left| a^{\mathrm{i}}_{F(f)} \right|^2 \left(\sum\nolimits_{k \neq m}^{n} \Delta \mathrm{t}_k \Delta \mathrm{t}_m \left\langle A_z(s_k) e^{\mathrm{i}}_{s_k, f(s_k)}, e^{\mathrm{i}}_{s_k, f(s_k)} \right\rangle \left\langle e^{\mathrm{i}}_{s_m, f(s_m)}, A_z(s_m) e^{\mathrm{i}}_{s_m, f(s_m)} \right\rangle \right)$$

$$+ \sum\nolimits_{\mathrm{i}}^{J} \sum\nolimits_{F(f)}^{K} \left| a^{\mathrm{i}}_{F(f)} \right|^2 \left(\sum\nolimits_{k=1}^{n} (\Delta \mathrm{t}_k)^2 \left\langle A_z(s_k) e^{\mathrm{i}}_{s_k, f(s_k)}, A_z(s_k) e^{\mathrm{i}}_{s_k, f(s_k)} \right\rangle \right).$$

This can be rewritten as

$$\begin{aligned} &\|\mathbf{Q}_{z,n}\Phi\|_{\otimes}^2 = \sum\nolimits_{i}^{J} \sum\nolimits_{F(f)}^{K} \left| a^{i}_{F(f)} \right|^2 \left\{ \left| \left\langle Q_{z,n} e^i, e^i \right\rangle \right|^2 \right. \\ (3.5)\quad & \left. + \sum\nolimits_{k=1}^{n} (\Delta t_k)^2 \left(\left\| A_z(s_k) e^i \right\|^2 - \left| \left\langle A_z(s_k) e^i, e^i \right\rangle \right|^2 \right) \right\}. \quad (a.s.c) \end{aligned}$$

First note that: $\left\| A_z(s_k) e^i \right\|^2 - \left| \left\langle A_z(s_k) e^i, e^i \right\rangle \right|^2 = \left\| A_z(s_k) e^i - \left\langle A_z(s_k) e^i, e^i \right\rangle e^i \right\|^2$, so that the last term in (3.6) can be written as

$$\sum\nolimits_{k=1}^{n} (\Delta t_k)^2 \left(\left\| A_z(s_k) e^i \right\|^2 - \left| \left\langle A_z(s_k) e^i, e^i \right\rangle \right|^2 \right) = \sum\nolimits_{k=1}^{n} (\Delta t_k)^2 \left\| A_z(s_k) e^i - \left\langle A_z(s_k) e^i, e^i \right\rangle e^i \right\|^2$$

$$\leqslant \mu_n^{\delta} M.$$

We can now use the above in (3.6) to get

$$\|\mathbf{Q}_{z,n}\Phi\|_{\otimes}^2 \leqslant \sum\nolimits_{i}^{J} \sum\nolimits_{F(f)}^{K} \left| a^{i}_{F(f)} \right|^2 \left| \left\langle Q_{z,n} e^i, e^i \right\rangle \right|^2 + \mu_n^{\delta} M \quad (a.s.c).$$

Thus, $\mathbf{Q}_{z,n}[t,a]$ converges strongly to $\mathbf{Q}_z[t,a]$ on $\mathcal{FD}_{\otimes}^2$. To show that $\mathbf{Q}_z[t,a]$ generates a uniformly continuous contraction semigroup, it suffices to show that $Q_{\mathrm{z}}[t,a]$ is dissipative. For any Φ in $\mathcal{FD}_{\otimes}^2$,

$$\langle \mathbf{Q}_z[t,a]\Phi, \Phi\rangle = \sum_{i}^{J} \sum_{F(f)}^{K} \left| a^{i}_{F(f)} \right|^2 \left\langle Q_z e^i, e^i \right\rangle \ (a.s.c)$$

and, for each n, we have

$$\operatorname{Re}\left\langle Q_z[t,a]e^i, e^i\right\rangle = \operatorname{Re}\left\langle Q_{z,n}[t,a]e^i, e^i\right\rangle + \operatorname{Re}\left\langle \left[Q_z[t,a] - Q_{z,n}[t,a]\right]e^i, e^i\right\rangle$$

$$\leqslant \operatorname{Re}\left\langle \left[Q_z[t,a] - Q_{z,n}[t,a]\right]e^i, e^i\right\rangle,$$

since $Q_{z,n}[t,a]$. Letting $n \to \infty$, $\Rightarrow$ $\operatorname{Re}\left\langle Q_z[t,a]e^i, e^i\right\rangle \leqslant 0$, so that $\operatorname{Re}\left\langle \mathbf{Q}_z[t,a]\Phi, \Phi\right\rangle \leqslant 0$. Thus, $\mathbf{Q}_z[t,a]$ is a bounded dissipative linear operator on $\mathcal{FD}_\otimes^2$ which completes our proof. $\square$

We can also prove Theorem 51 for the family $\{\mathcal{A}(\mathrm{t}) \mid t \in I\}$. The same proof goes through, but now we restrict to $\mathrm{D}_0 = \underset{t\in I}{\otimes} \mathrm{D}(A(t)) \cap \mathcal{FD}_\otimes^2$. In this case (3.4) becomes:

$$\sum_{k,}^{n} \Delta t_k \left\| A(s_k)e^i - \left\langle A(s_k)e^i, e^i\right\rangle e^i \right\|^2 \leqslant M\mu_n^{\delta-1}. \tag{3.6}$$

From equation (3.6), we have the following important result: (set $\sum\limits_{F(f)}^{K} \left| a_{F(f)}^{\mathrm{i}} \right|^2 = \left| b^i \right|^2$)

$$\left\| \mathbf{Q}_z[t,a]\Phi \right\|_\otimes^2 = \sum_{\mathrm{i}}^{J} \left| b^i \right|^2 \left| \left\langle Q_z e^i, e^i\right\rangle \right|^2 \ (a.s.c). \tag{3.7}$$

The representation (3.7) makes it easy to prove the next theorem.

Theorem 53. *With the conditions of Theorem 51, we have:*

(1) $\mathbf{Q}_z[t,s] + \mathbf{Q}_z[s,a] = \mathbf{Q}_z[t,a]$ *(a.s.c)*,

(2) s - $\lim\limits_{h\to 0} \frac{\mathbf{Q}_z[t+h,a] - \mathbf{Q}_z[t,a]}{h} = s$ - $\lim\limits_{h\to 0} \frac{\mathbf{Q}_z[t+h,t]}{h} = \mathcal{A}_z(t)$ *(a.s.c)*,

(3) s - $\lim\limits_{h\to 0} \mathbf{Q}_z[t+h,t] = 0$ *(a.s.c)*,

(4) s - $\lim\limits_{h\to 0} \exp\left\{\tau \mathbf{Q}_z[t+h,t]\right\} = I_\otimes$ *(a.s.c)*, $\tau \geqslant 0$.

Proof. In each case, it suffices to prove the result for $\Phi \in \mathrm{D}_0$. To prove 1., use

$$\left\| [\mathbf{Q}_z[t,s] + \mathbf{Q}_z[s,a]]\, \Phi \right\|_{\otimes}^2 = \sum\nolimits_i^J \left| b^i \right|^2 \left| \left\langle [Q_z[t,s] + Q_z[s,a]]\, e^i, e^i \right\rangle \right|^2$$

$$= \sum\nolimits_i^J \left| b^i \right|^2 \left| \left\langle Q_z[t,a] e^i, e^i \right\rangle \right|^2 = \left\| \mathbf{Q}_z[t,a] \Phi \right\|_{\otimes}^2 \text{ (a.s.c).}$$

To prove 2., use 1. to get that $\mathbf{Q}_z[t+h,a] - \mathbf{Q}_z[t,a] = \mathbf{Q}_z[t+h,t]$ (a.s.), so that

$$\lim_{h \to 0} \left\| \frac{\mathbf{Q}_z[t+h,t]}{h} \Phi \right\|_{\otimes}^2$$

$$= \sum_{\mathrm{i}}^{J} \left| b^i \right|^2 \lim_{h \to 0} \left| \left\langle \frac{Q_z[t+h,t]}{h} e^i, e^i \right\rangle \right|^2 = \left\| \mathcal{A}_z(t) \Phi \right\|_{\otimes}^2 \text{ (a.s.c.).}$$

The proof of 3 follows from 2 and the proof of 4 follows from 3. $\square$

The results of the previous theorem are expected if $\mathbf{Q}_z[t,a]$ is an integral in the conventional sense. The important point is that a weak integral on the base space gives a strong integral on $\mathcal{FD}_{\otimes}^2$ (note that by 2., we also get strong differentiability). This clearly shows that our approach to time ordering has more to offer than simply a representation space to allow time to act as a place keeper for operators in a product. It should be observed that, in all results up to now, we have used the assumption that the family $A(t), t \in I$, is weakly continuous, satisfies equation (3.6), and has a common dense domain $D \subseteq D(A(t))$ in $\mathcal{H}$. We now impose a condition that is equivalent to assuming that each $A(t)$ generates a C_0-contraction semigroup; namely, we assume that, for each t, $A(t)$ and $A^*(t)$ (dual) are dissipative. This form is an easier condition to check.

Theorem 54. *With the above assumptions, we have that* $\lim_{z\to\infty}\langle Q_z[t,a]\phi,\psi\rangle = \langle Q[t,a]\phi,\psi\rangle$ *exists for all* $\phi \in D[Q]$, $\psi \in D[Q^*]$. *Furthermore:*

(1) *the operator* $Q[t,a]$ *generates a* C_0*-contraction semigroup on* $\mathcal{H}$*,*

(2) *for* $\Phi \in D_0$*,*

$$\lim_{z\to\infty}\mathbf{Q}_z[t,a]\Phi = \mathbf{Q}[t,a]\Phi,$$

and

(3) *the operator* $\mathbf{Q}[t,a]$ *generates a* C_0*-contraction semigroup on* $\mathcal{FD}^2_{\otimes}$*,*

(4) $\mathbf{Q}[t,s]\Phi + \mathbf{Q}[s,a]\Phi = \mathbf{Q}[t,a]\Phi$ *(a.s.c.),*

(5)

$$\lim_{h\to 0}\left[(\mathbf{Q}[t+h,a]-\mathbf{Q}[t,a])/h\right]\Phi = \lim_{h\to 0}\left[(\mathbf{Q}[t+h,t])/h\right]\Phi = \mathcal{A}(t)\Phi \text{ (a.s.c.)},$$

(6) $\lim_{h\to 0}\mathbf{Q}[t+h,t]\Phi = 0$ *(a.s.c.), and*

(7) $\lim_{h\to 0}\exp\{\tau\mathbf{Q}[t+h,t]\}\Phi = \Phi$ *(a.s.c.),*$\tau \geqslant 0$.

Proof. Since $A_z(t)$, $A(t)$ are weakly continuous and $A_z(t) \xrightarrow{s} A(t)$ for each $t \in I$, given $\varepsilon > 0$ we can choose Z such that, if $z > Z$, then

$$\sup_{s\in[a,b]}|\langle[A(s)-A_z(s)]\,\varphi\,,\,\psi\rangle| < \varepsilon/3(b-a).$$

By uniform (weak) continuity, if $s, s' \in [a,b]$ we can also choose η such that, if $|s-s'| < \eta$,

$$\sup_{z>0}\left|\left\langle\left[A_z(s)-A_z(s')\right]\varphi\,,\,\psi\right\rangle\right| < \varepsilon/3(b-a)$$

and

$$\left|\left\langle \left[A(s) - A(s')\right] \varphi\, ,\, \psi \right\rangle\right| < \varepsilon/3(b-a).$$

Now choose $\delta(t)\, :\ [a,b] \to (0,\infty)$ so that, for any KH-partition $\mathbf{P}$ for δ, we have that $\mu_n < \eta$, where μ_n is the mesh of the partition. If $Q_{z,n} = \sum_{j=1}^{n} A_z(\tau_j)\Delta t_j$ and $Q_n = \sum_{j=1}^{n} A(\tau_j)\Delta t_j$, we have

$$\begin{aligned}
&|\langle [Q_z[t,a] - Q[t,a]]\, \varphi\, ,\, \psi\rangle| \leqslant |\langle [Q_n[t,a] - Q[t,a]]\, \varphi\, ,\, \psi\rangle| \\
&\quad + |\langle [Q_{z,n}[t,a] - Q_z[t,a]]\, \varphi\, ,\, \psi\rangle| + |\langle [Q_n[t,a] - Q_{z,n}[t,a]]\, \varphi\, ,\, \psi\rangle| \\
&\quad \leqslant \sum\nolimits_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\langle [A(\tau_j) - A(\tau)]\, \varphi\, ,\, \psi\rangle|\, d\tau + \sum\nolimits_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\langle [A_z(\tau_j) - A_z(\tau)]\, \varphi\, ,\, \psi\rangle|\, d\tau \\
&\quad + \sum\nolimits_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\langle [A(\tau_j) - A_z(\tau_j)]\, \varphi\, ,\, \psi\rangle|\, d\tau < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This proves that $\lim\limits_{z\to\infty} \langle Q_z[t,a]\phi, \psi\rangle = \langle Q[t,a]\phi, \psi\rangle$. To prove 1., first note that $Q[t,a]$ is closable and use

$$\begin{aligned}
&\operatorname{Re}\langle Q[t,a]\phi, \phi\rangle = \operatorname{Re}\langle Q_z[t,a]\phi, \phi\rangle + \operatorname{Re}\langle [Q[t,a] - Q_z[t,a]]\, \phi, \phi\rangle \\
&\quad \leqslant \operatorname{Re}\langle [Q[t,a] - Q_z[t,a]]\, \phi, \phi\rangle\, ,
\end{aligned}$$

and let z $\to \infty$, to show that $Q[t,a]$ is dissapative. Then do likewise for $\langle \phi, Q^*[t,a]\phi\rangle$ to show that the same is true for $Q^*[t,a]$, to complete the proof. (*It is important to note that, although* $Q[t,a]$ *generates a contraction semigroup on* $\mathcal{H}$*, exp*$\{Q[t,a]\}$ *does not solve the original initial-value problem.*)

To prove (2), use (3.7) in the form

$$(3.8)\|\left[\mathbf{Q}_z[t,a] - \mathbf{Q}_{z'}[t,a]\right]\Phi\|_{\otimes}^2 = \sum_{\text{i}}^{J} \left|b^i\right|^2 \left|\left\langle [Q_z[t,a] - Q_{z'}[t,a]]\, e^i, e^i\right\rangle\right|^2 .$$

This proves that $\mathbf{Q}_z[t,a] \xrightarrow{\text{s}} \mathbf{Q}[t,a]$. Since $\mathbf{Q}[t,a]$ is densely defined, it is closable. The same method as above shows that it is m-dissipative. Proofs of the other results follow the methods of Theorem 54. $\square$

3.4. **General Case.**

. We relax the contraction condition and assume that $A(t), t \in I$ generates a C_0-semigroup on $\mathcal{H}$. We can always shift the spectrum (if necessary) so that $\|\exp\{\tau A(t)\}\| \leqslant M(t)$. We assume that $\sup_J \prod_{i \in J} \|\exp\{\tau A(t_i)\}\| \leqslant M$, where the sup is over all finite subsets $J \subset I$.

Theorem 55. *Suppose that $A(t), t \in I$, generates a C_0-semigroup, satisfies (3.6) and has a weak KH-integral, $Q[t,a]$, on a dense set D in $\mathcal{H}$. Then the family $\mathcal{A}(t), t \in I$, has a strong KH-integral, $\mathbf{Q}[t,a]$, which generates a C_0-semigroup on $\mathcal{FD}_\otimes^2$ (for each $t \in I$) and $\|\exp\{\mathbf{Q}[t,a]\}\|_\otimes \leqslant M$.*

Proof. It is clear from part (2) of Theorem 51 that $\mathbf{Q}_n[t,a] = \sum_{i=1}^n \mathcal{A}(\tau_i)\Delta t_i$ generates a C_0-semigroup on $\mathcal{FD}_\otimes^2$ and $\|\exp\{\mathbf{Q}_n[t,a]\}\|_\otimes \leqslant M$. If $\Phi \in D_0$, let $\mathbf{P}_m$, $\mathbf{P}_n$ be arbitrary KH-partitions for δ_m, δ_n (of order m and n respectively) and set $\delta(s) = \delta_m(s) \wedge \delta_n(s)$. Since any KH-partition for δ is

one for δ_m and δ_n, we have that

$$\begin{aligned}
&\left\| \left[\exp\{\tau\mathbf{Q}_n[t,a]\} - \exp\{\tau\mathbf{Q}_m[t,a]\}\right]\Phi\right\|_\otimes \\
&\quad = \left\| \int_0^\tau \frac{d}{ds}\left[\exp\{(\tau - s)\mathbf{Q}_n[t,a]\}\exp\{s\mathbf{Q}_m[t,a]\}\right]\Phi ds \right\|_\otimes \\
&\quad \leqslant \int_0^\tau \left\| \left[\exp\{(\tau - s)\mathbf{Q}_n[t,a]\}\left(\mathbf{Q}_n[t,a] - \mathbf{Q}_m[t,a]\right)\exp\{s\mathbf{Q}_m[t,a]\}\Phi\right]\right\|_\otimes \\
&\quad \leqslant M\int_0^\tau \left\| \left(\mathbf{Q}_n[t,a] - \mathbf{Q}_m[t,a]\right)\Phi\right\|_\otimes ds \\
&\quad \leqslant M\tau \left\| \left[\mathbf{Q}_n[t,a] - \mathbf{Q}[t,a]\right]\Phi\right\|_\otimes + M\tau \left\| \left[\mathbf{Q}[t,a] - \mathbf{Q}_m[t,a]\right]\Phi\right\|_\otimes .
\end{aligned}$$

The existence of the weak KH-integral, $Q[t,a]$, on $\mathcal{H}$ satisfying equation (3.6) implies that $\mathbf{Q}_n[t,a] \xrightarrow{s} \mathbf{Q}[t,a]$, so that $\exp\{\tau\mathbf{Q}_n[t,a]\}\Phi$ converges as $n \to \infty$ for each fixed $t \in I$; and the convergence is uniform on bounded τ intervals. As $\|\exp\{\mathbf{Q}_n[t,a]\}\|_\otimes \leqslant M$, we have

$$\lim_{n\to\infty} \exp\{\tau\mathbf{Q}_n[t,a]\}\Phi = \mathbf{S}_t(\tau)\Phi,\ \ \Phi \in \mathcal{FD}_\otimes^2.$$

The limit is again uniform on bounded τ intervals. It is easy to see that the limit $\mathbf{S}_t(\tau)$ satisfies the semigroup property, $\mathbf{S}_t(0) = I$, and $\|\mathbf{S}_t(\tau)\|_\otimes \leqslant M$. Furthermore, as the uniform limit of continuous functions, we see that $\tau \to \mathbf{S}_t(\tau)\Phi$ is continuous for $\tau \geqslant 0$. We are done if we show that $\mathbf{Q}[t,a]$ is the generator of $\mathbf{S}_t(\tau)$. For $\Phi \in D_0$, we have that

$$\begin{aligned}
\mathbf{S}_t(\tau)\Phi - \Phi &= \lim_{n\to\infty} \exp\{\tau\mathbf{Q}_n[t,a]\}\Phi - \Phi \\
&= \lim_{n\to\infty} \int_0^\tau \exp\{s\mathbf{Q}_n[t,a]\}\mathbf{Q}_n[t,a]\Phi ds = \int_0^\tau \mathbf{S}_t(\tau)\mathbf{Q}[t,a]\Phi ds.
\end{aligned}$$

Our result follows from the uniqueness of the generator, so that $\mathbf{S}_t(\tau) = \exp\{\tau\mathbf{Q}[t,a]\}$. □

The next result is the time-ordered version of the Hille-Yosida Theorem (see Pazy [PZ], pg. 8). We assume that the family $A(t), t \in I$, is closed and densely defined.

Theorem 56. *The family $\mathcal{A}(t), t \in I$, has a strong KH-integral, $\mathbf{Q}[t,a]$, which generates a C_0-contraction semigroup on $\mathcal{FD}^2_{\otimes}$ if and only if $\rho(A(t)) \supset (0,\infty)$, $\|R(\lambda : A(t))\| < 1/\lambda$, for $\lambda > 0$, $A(t), t \in I$ satisfies (3.6) and has a densely defined weak KH-integral $Q[t,a]$ on $\mathcal{H}$.*

Proof. In the first direction, suppose $\mathbf{Q}[t,a]$ generates a C_0-contraction semigroup on $\mathcal{FD}^2_{\otimes}$, then $\mathbf{Q}_n[t,a]\Phi \xrightarrow{s} \mathbf{Q}[t,a]\Phi$ for each $\Phi \in \mathrm{D}_0$, and each $t \in I$. Since $\mathbf{Q}[t,a]$ has a densely defined strong KH-integral, it follows from (3.6) that $Q[t,a]$ must have a densely defined weak KH-integral. Since $\mathbf{Q}_n[t,a]$ generates a C_0-contraction semigroup for each KH-partition of order n, it follows that $\mathcal{A}(t)$ must generate a C_0-contraction semigroup for each $t \in I$. From Theorem 42 and Theorem 51, we see that $A(t)$ must also generate a C_0-contraction semigroup for each $t \in I$. From the conventional Hille-Yosida theorem, the resolvent condition follows.

In the reverse direction, the conventional Hille-Yosida theorem along with the first part of Theorem 55 shows that $Q[t,a]$ generates a C_0-contraction semigroup for each $t \in I$. From parts 2, 3 of Theorem 51 and Theorem 42, we have that for each KH-partition of order n, $\mathbf{Q}_n[t,a]$ generates a C_0-contraction semigroup, $\mathbf{Q}_{\mathrm{n}}[t,a]\Phi \to \mathbf{Q}[t,a]\Phi$ for each $\Phi \in \mathrm{D}_0$ and each $t \in I$, and $\mathbf{Q}[t,a]$ generates a C_0-contraction semigroup on $\mathcal{FD}^2_{\otimes}$. $\square$

The other generation theorems have a corresponding formulation in terms of time-ordered integrals.

4. **Time-Ordered Evolutions**

As $\mathbf{Q}[t,a]$ and $\mathbf{Q}_{\mathrm{z}}[t,a]$ generate (uniformly bounded) C_0-semigroups, we can set $\mathbf{U}[t,a] = exp\{\mathbf{Q}[t,a]\}$, $\mathbf{U}_z[t,a] = exp\{\mathbf{Q}_z[t,a]\}$. They are C_0-evolution operators and the following theorem generalizes a result due to Hille and Phillips [HP].

Theorem 57. *For each n, and $\Phi \in D\left[(\mathbf{Q}[t,a])^{n+1}\right]$, we have: ($w$ is positive and $\mathbf{U}^w[t,a] = exp\left\{w\mathbf{Q}[t,a]\right\}$)*

$$\mathbf{U}^w[t,a]\Phi = \left\{ I_\otimes + \sum_{k=1}^{n} \frac{(w\mathbf{Q}[t,a])^n}{n!} + \frac{1}{n!}\int\limits_0^w (w-\xi)^n \mathbf{Q}[t,a]^{n+1}\mathbf{U}^\xi[t,a]d\xi \right\}\Phi.$$

Proof. The proof is easy, start with

$$\left[\mathbf{U}_{\mathrm{z}}^w[t,a]\Phi - I_\otimes\right]\Phi = \int\limits_0^w \mathbf{Q}_{\mathrm{z}}[t,a]\mathbf{U}_{\mathrm{z}}^\xi[t,a]d\xi\Phi$$

and use integration by parts to get that

$$\left[\mathbf{U}_z^w[t,a]\Phi - I_\otimes\right]\Phi = w\mathbf{Q}_z[t,a]\Phi + \int\limits_0^w (w-\xi)\left[\mathbf{Q}_{\mathrm{z}}[t,a]\right]^2 \mathbf{U}_z^\xi[t,a]d\xi\Phi.$$

It is clear how to get the nth term. Finally, let $z \to \infty$ to get the result. $\square$

Theorem 58. *If $a < t < b$,*

(1) $\lim\limits_{z\to\infty} \mathbf{U}_z[t,a]\Phi = \mathbf{U}[t,a]\Phi,\ \Phi \in \mathcal{FD}_\otimes^2.$

(2)

$$\frac{\partial}{\partial t}\mathbf{U}_z[t,a]\Phi = \mathcal{A}_z(t)\mathbf{U}_z[t,a]\Phi = \mathbf{U}_z[t,a]\mathcal{A}_z(t)\Phi,$$

with $\Phi \in \mathcal{FD}_\otimes^2$, *and*

(3)

$$\frac{\partial}{\partial t}\mathbf{U}[t,a]\Phi = \mathcal{A}(t)\mathbf{U}[t,a]\Phi = \mathbf{U}[t,a]\mathcal{A}(t)\Phi,\ \Phi \in D(\mathbf{Q}[b,a]) \supset D_0.$$

Proof. To prove (1), use the fact that $\mathcal{A}_z(t)$ and $\mathcal{A}(t)$ commute, along with

$$\mathbf{U}[t,a]\Phi - \mathbf{U}_z[t,a]\Phi = \int_0^1 (d/ds)\left(e^{s\mathbf{Q}[t,a]}e^{(1-s)\mathbf{Q}_z[t,a]}\right)\Phi ds$$
$$= \int_0^1 s\left(e^{s\mathbf{Q}[t,a]}e^{(1-s)\mathbf{Q}_z[t,a]}\right)(\mathbf{Q}[t,a] - \mathbf{Q}_z[t,a])\,\Phi ds,$$

so that

$$\lim_{z\to 0}\|\mathbf{U}[t,a]\Phi - \mathbf{U}_z[t,a]\Phi\| \leqslant M\lim_{z\to 0}\|\mathbf{Q}[t,a]\Phi - \mathbf{Q}_z[t,a]\Phi\| = 0.$$

To prove (2), use

$$\mathbf{U}_z[t+h,a]-\mathbf{U}_z[t,a] = \mathbf{U}_z[t,a]\left(\mathbf{U}_z[t+h,t]-\mathrm{I}\right) = \left(\mathbf{U}_z[t+h,t]-\mathrm{I}\right)\mathbf{U}_z[t,a],,$$

so that

$$(\mathbf{U}_z[t+h,a] - \mathbf{U}_z[t,a])/h = \mathbf{U}_z[t,a]\left[(\mathbf{U}_z[t+h,t]-\mathrm{I})/h\right].$$

Now set $\Phi_z^t = \mathbf{U}_z[t,a]\Phi$ and use Theorem 58 with $n=1$ and $w=1$ to get:

$$\mathbf{U}_z[t+h,t]\Phi_z^t = \left\{I_\otimes + \mathbf{Q}_z[t+h,t] + \int\limits_0^1 (1-\xi)\mathbf{U}_z^\xi[t+h,t]\mathbf{Q}_z[t+h,t]^2 d\xi\right\}\Phi_z^t,$$

so

$$\frac{(\mathbf{U}_z[t+h,t]-\mathrm{I})}{h}\Phi_z^t-\mathcal{A}_z(t)\Phi_z^t=\frac{\mathbf{Q}_z[t+h,t]}{h}\Phi_z^t-\mathcal{A}_z(t)\Phi_z^t$$
$$+\int\limits_0^1(1-\xi)\mathbf{U}_z^{\xi}[t+h,t]\frac{\mathbf{Q}_z[t+h,t]}{h}^2\Phi_z^t d\xi.$$

It follows that

$$\left\|\frac{(\mathbf{U}_z[t+h,t]-\mathrm{I})}{h}\Phi_z^t-\mathcal{A}_z(t)\Phi_z^t\right\|_{\otimes}\leqslant\left\|\frac{\mathbf{Q}_z[t+h,t]}{h}\Phi_z^t-\mathcal{A}_z(t)\Phi_z^t\right\|_{\otimes}+\frac{1}{2}\left\|\frac{\mathbf{Q}_z[t+h,t]}{h}^2\Phi_z^t\right\|_{\otimes}.$$

The result now follows from Theorem 54, (2) and (3) To prove (3), note that $\mathcal{A}_z(t)\Phi=\mathcal{A}(t)\left\{z\mathbf{R}(z,\mathcal{A}(t))\right\}\Phi=\left\{z\mathbf{R}(z,\mathcal{A}(t))\right\}\mathcal{A}(t)\Phi$, so that $\{z\mathbf{R}(z,\mathcal{A}(t))\}$ commutes with $\mathbf{U}[t,a]$ and $\mathcal{A}(t)$. It is now easy to show that

$$\begin{aligned}
&\|\mathcal{A}_z(t)\mathbf{U}_z[t,a]\Phi-\mathcal{A}_{z'}(t)\mathbf{U}_{z'}[t,a]\Phi\|\\
&\quad\leqslant\|\mathbf{U}_z[t,a]\left(\mathcal{A}_z(t)-\mathcal{A}_{z'}(t)\right)\Phi\|+\left\|z'\mathbf{R}(z',\mathcal{A}(t))\left[\mathbf{U}_z[t,a]\Phi-\mathbf{U}_{z'}[t,a]\right]\mathcal{A}(t)\Phi\right\|\\
&\quad\leqslant M\left\|\left(\mathcal{A}_z(t)-\mathcal{A}_{z'}(t)\right)\Phi\right\|+M\left\|\left[\mathbf{U}_z[t,a]\Phi-\mathbf{U}_{z'}[t,a]\right]\mathcal{A}(t)\Phi\right\|\to 0,\ z,z'\to\infty,
\end{aligned}$$

so that, for $\Phi\in\mathrm{D}(\mathbf{Q}[b,a])$,

$$\mathcal{A}_z(t)\mathbf{U}_z[t,a]\Phi\to\mathcal{A}(t)\mathbf{U}[t,a]\Phi=\frac{\partial}{\partial t}\mathbf{U}[t,a]\Phi.$$

$\square$

Since, as noted earlier, $exp\{Q[t,a]\}$ does not solve the initial-value problem, we restate the last part of the last theorem to emphasize the importance of this result, and the power of the constructive Feynman theory.

Theorem 59. *If $a < t < b$,*

$$\frac{\partial}{\partial t}\mathbf{U}[t,a]\Phi = \mathcal{A}(t)\mathbf{U}[t,a]\Phi = \mathbf{U}[t,a]\mathcal{A}(t)\Phi,\ \Phi \in D_0 \subset D(\mathbf{Q}[b,a]).$$

4.1. **Application: Hyperbolic and Parabolic Evolution Equations.**

. We can now apply the previous results to show that the standard conditions imposed in the study of hyperbolic and parabolic evolution equations imply that the family of operators is strongly continuous (see Pazy [PZ]), so that our condition (3.6) is automatically satisfied. Let us recall the specific assumptions traditionally assumed in the study of parabolic and hyperbolic evolution equations. Without loss, we shift the spectrum of $A(t)$ at each t, if necessary, to obtain a uniformly bounded family of semigroups.

Parabolic Case

In the abstract approach to parabolic evolution equations, it is assumed that:

(1) For each $t \in I,\ A(t)$ generates an analytic C_0-semigroup with domains $D(A(t)) = D$ independent of t.

(2) For each $t \in I,\ R(\lambda, A(t))$ exists $\forall \lambda \ni\ \operatorname{Re}\lambda \leqslant 0$, and there is an $M > 0$ such that:

$$\|R(\lambda, A(t))\| \leqslant M/[|\lambda| + 1].$$

(3) There exist constants L and $0 < \alpha \leqslant 1$ such that

$$\left\|\left(A(t) - A(s)\right)A(\tau)^{-1}\right\| \leqslant L\,|t - s|^{\alpha} \quad \forall\, t,\ s,\ \tau \in I$$

In this case, when (3) is satisfied and $\varphi \in D$, we have

$$\|[A(t) - A(s)]\,\varphi\| = \left\|\left[(A(t) - A(s))\,A^{-1}(\tau)\right]A(\tau)\varphi\right\|$$

$$\leqslant \left\|(A(t) - A(s))\,A^{-1}(\tau)\right\|\|A(\tau)\varphi\| \leqslant L\,|t - s|^{\alpha}\,\|A(\tau)\varphi\|\,,$$

so that the family $A(t),\ t \in I$, is strongly continuous on D. It follows that the time ordered family $\mathcal{A}(t),\ t \in I$, has a strong Riemann integral on D_0.

Hyperbolic Case

In the abstract approach to hyperbolic evolution equations, it is assumed that:

(1) For each $t \in I,\ A(t)$ generates a C_0-semigroup.

(2) For each $t,\ A(t)$ is stable with constants $M,\ 0$ and $\rho(A(t)) \supset (0, \infty),\ t \in I$ (the resolvent set for $A(t)$), such that:

$$\left\|\prod_{j=1}^{k}\exp\{\tau_j A(t_j)\}\right\| \leqslant M.$$

(3) There exists a Hilbert space $\mathcal{Y}$ densely and continuously embedded in $\mathcal{H}$ such that, for each $t \in I,\ D(A(t)) \supset \mathcal{Y}$ and $A(t) \in L[\mathcal{Y}, \mathcal{H}]$ (i.e., $A(t)$ is bounded as a mapping from $\mathcal{Y} \to \mathcal{H}$), and the function $g(t) = \|A(t)\|_{\mathcal{Y}\to\mathcal{H}}$ is continuous.

(4) The space $\mathcal{Y}$ is an invariant subspace for each semigroup $S_t(\tau) = \exp\{\tau A(t)\}$ and $S_t(\tau)$ is a stable C_0-semigroup on $\mathcal{Y}$ with the same stability constants.

This case is not as easily analyzed as the parabolic case, so we need the following:

Lemma 60. *Suppose conditions (3) and (4) above are satisfied with* $\|\varphi\|_{\mathcal{H}} \leqslant \|\varphi\|_{\mathcal{Y}}$*. Then the family* $A(t),\ t \in I$ *is strongly continuous on* $\mathcal{H}$*.*

Proof. Let $\varepsilon > 0$ be given and, without loss, assume that $\|\varphi\|_{\mathcal{H}} \leqslant 1$. Set $c = \|\varphi\|_{\mathcal{Y}} / \|\varphi\|_{\mathcal{H}}$, so that $1 \leqslant c < \infty$. Now

$$\|[A(t+h) - A(t)]\,\varphi\|_{\mathcal{H}} \leqslant \left\{ \|[A(t+h) - A(t)]\,\varphi\|_{\mathcal{H}} / \|\varphi\|_{\mathcal{Y}} \right\} \left[\|\varphi\|_{\mathcal{Y}} / \|\varphi\|_{\mathcal{H}} \right]$$

$$\leqslant c\,\|A(t+h) - A(t)\|_{\mathcal{Y}\to\mathcal{H}}\,.$$

Choose $\delta > 0$ such that $|h| < \delta \Rightarrow \|A(t+h) - A(t)\|_{\mathcal{Y}\to\mathcal{H}} < \varepsilon/c$, which completes the proof. $\square$

5. **Perturbation Theory**

The study of perturbation theory for semigroups of operators has two different approaches. Both have their roots in mathematical physics however the first is motivated by and concerned with problems of modern physics, while the concerns of the second approach has moved to the larger domain of functional analysis, partial differential equations and applied mathematics. In this section, we prove a few results for both types without attempting to be exhaustive, since the known results (of perturbation theory) have a direct extension to the time-ordered setting. Because of Theorem 27, the general problem of perturbation theory can always be reduced to that of the strong limit of the bounded case. Assume that, for each $t \in I$, $A_0(t)$ is the generator of a C_0-semigroup on $\mathcal{H}$ and that $A_1(t)$ is closed and densely defined. The (generalized) sum of $A_0(t)$ and $A_1(t)$, in its various forms, whenever it is

defined (with dense domain), is denoted by $A(t) = A_0(t) \oplus A_1(t)$ (see Kato [KA], and Pazy [PZ]). Let $A_1^n(t) = nA_1(t)R(n, T_1(t))$ be the (generalized) Yosida approximator for $A_1(t)$, where $T_1(t) = -\left[A_1^*(t)A_1(t)\right]^{1/2}$ and set $A_n(t) = A_0(t) + A_1^n(t)$.

Theorem 61. *For each n, $A_0(t) + A_1^n(t)$ (respectively $\mathcal{A}_0(t) + \mathcal{A}_1^n(t)$) is the generator of a C_0-semigroup on $\mathcal{H}$ (respectively $\mathcal{FD}_\otimes^2$) and:*

(1) *If for each $t \in I$, $A_0(t)$ generates an analytic or contraction C_0-semigroup then so does $A_n(t)$ and $\mathcal{A}_n(t)$.*

(2) *If for each $t \in I$, $A(t) = A_0(t) \oplus A_1(t)$ generates an analytic or contraction C_0-semigroup, then so does $\mathcal{A}(t) = \mathcal{A}_0(t) \oplus \mathcal{A}_1(t)$ and $\exp\{\tau\mathcal{A}_n(t)\} \to \exp\{\tau\mathcal{A}(t)\}$ for $\tau \geqslant 0$.*

Proof. The first two parts of (1) are standard (see Pazy [PZ] pg. 79, 81). The third part (contraction) follows because $A_1^n(t)$ (respectively $\mathcal{A}_1^n(t)$) is a bounded m-dissipative operator. The proof of (2) follows from Theorem 27 equation (3.3) and Theorem 42. □

We now assume that $A_0(t)$ and $A_1(t)$ are weakly continuous, generators of C_0-semigroups for each $t \in I$, and equation (3.6) is satisfied. Then, with the same notation, we have:

Theorem 62. *If, for each $t \in I$, $A(t) = A_0(t) \oplus A_1(t)$ generates an analytic or contraction semigroup, then $\mathbf{Q}[t, a]$ generates an analytic or contraction semigroup and $\exp\{\mathbf{Q}_n[t, a]\} \to \exp\{\mathbf{Q}[t, a]\}$.*

Proof. The proof follows from Theorem 52 and Theorem 56. □

5.1. **Interaction Representation.**

. The physical research related to this paper is part of a different point of departure in the investigation of the foundations of relativistic quantum theory (compared to axiomatic or constructive field theory approaches) and therefore considers different problems and questions (see [GJ] and also [SW]). However, within the framework of axiomatic field theory, an important theorem of Haag suggests that the interaction representation, used in theoretical physics, does not exist in a rigorous sense (see Streater and Wightman, [SW] pg. 161). Haag's theorem shows that the equal time commutation relations for the canonical variables of an interacting field are equivalent to those of a free field. In trying to explain this unfortunate result, Streater and Wightman point out that (see p. 168) "... What is even more likely in physically interesting quantum field theories is that equal time commutation relations will make no sense at all; the field might not be an operator unless smeared in time as well as space." In this section, it is first shown that, if one assumes (as Haag did) that operators act in sharp time, then the interaction representation (essentially) does not exist.

We know from elementary quantum theory that there is some overlapping of wave packets, so that it is more natural to expect smearing in time. In fact, striking results of a beautiful recent experiment of Lindner et al

(see Horwitz [HW] and references therein) clearly shows the effect of quantum interference in time for the wave function of a particle. Horwitz [HW] shows that the experiment has fundamental importance that goes beyond the technical advances the work of Lindner, et al represents, since a complete analysis requires relativistic quantum theory. In this section, we also show that, if any time smearing is allowed, then the interaction representation is well defined.

Let us assume that $A_0(t)$ and $A_1(t)$ are weakly continuous, generators of a C_0-unitary groups for each $t \in I$, $A(t) = A_0(t) \oplus A_1(t)$ is densely defined and equation (3.6) is satisfied. Define $\mathbf{U}_n[t,a]$, $\mathbf{U}_0[t,a]$ and $\bar{\mathbf{U}}_0^\sigma[t,a]$ by:

$$\mathbf{U}_n[t,a] = \exp\{(-i/\hbar)\int_a^t [\mathcal{A}_0(s) + \mathcal{A}_1^n(s)]ds\},$$

$$\mathbf{U}_0[t,a] = \exp\{(-i/\hbar)\int_a^t \mathcal{A}_0(s)ds\},$$

$$\bar{\mathbf{U}}_0[t,a] = \exp\{(-i/\hbar)\int_a^t \mathbf{E}[t,s]\mathcal{A}_0(s)ds\},$$

where $\mathbf{E}[t,s]$ is the standard exchange operator (see Definition 47 and Theorem 48). There are other possibilities, for example, we could replace $\bar{\mathbf{U}}_0[t,a]$ by $\bar{\mathbf{U}}_0^\sigma[t,a]$, where

$$\bar{\mathbf{U}}_0^\sigma[t,a] = \exp\{(-i/\hbar)\int_a^t \hat{\mathcal{A}}_0^\sigma(s)ds\},$$

$$\hat{\mathcal{A}}_0^\sigma(t) = \int_{-\infty}^{\infty} \rho_\sigma(t,s)\mathbf{E}[t,s]\mathcal{A}_0(s)ds,$$

where $\rho_\sigma(t,s)$ is a smearing density that may depend on a small parameter σ with $\int_{-\infty}^{\infty}\rho_\sigma(t,s)ds = 1$ (for example, $\rho_\sigma(t,s) = [1\Big/\sqrt{2\pi\sigma^2}]\exp\{-(t-s)^2/2\sigma^2\}$).

In the first case, using $\mathbf{U}_0[t,a]$, the interaction representation for $\mathcal{A}_1^n(t)$ is given by:

$$\mathcal{A}_{\mathbf{I}}^n(t) = \mathbf{U}_0[a,t]\mathcal{A}_1^n(t)\mathbf{U}_0[t,a] = \mathcal{A}_1^n(t), (a.s)$$

as $\mathcal{A}_1^n(t)$ commutes with $\mathbf{U}_0[a,t]$ in sharp time. Thus, the interaction representation does not exist. In either of the last two possibilities, we have

$$\mathcal{A}_{\mathbf{I}}^n(t) = \bar{\mathbf{U}}_0^\sigma[a,t]\mathcal{A}_1^n(t)\bar{\mathbf{U}}_0^\sigma[t,a],$$

and the terms do not commute. If we set $\Psi_n(t) = \bar{\mathbf{U}}_0^\sigma[a,t]\mathbf{U}_n[t,a]\Phi$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\Psi_n(t) = \frac{i}{\hbar}\bar{\mathbf{U}}_0^\sigma[a,t]\mathcal{A}_0(t)\mathbf{U}_n[t,a]\Phi - \frac{i}{\hbar}\bar{\mathbf{U}}_0^\sigma[a,t]\left[\mathcal{A}_0(t)+\mathcal{A}_1^n(t)\right]\mathbf{U}_n[t,a]\Phi \\
&\Rightarrow \frac{\partial}{\partial t}\Psi_n(t) = \frac{i}{\hbar}\{\bar{\mathbf{U}}_0^\sigma[a,t]\mathcal{A}_1^n(t)\bar{\mathbf{U}}_0^\sigma[t,a]\}\bar{\mathbf{U}}_0^\sigma[a,t]\mathbf{U}_n[t,a]\Phi \\
&\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi_n(t) = \mathcal{A}_{\mathbf{I}}^n(t)\Psi_n(t),\ \Psi_n(a) = \Phi.
\end{aligned}$$

With the same conditions as Theorem 62, we have

Theorem 63. *If* $Q_1[t,a] = \int_a^t A_1(s)ds$ *generates a* C_0*-unitary group on* $\mathcal{H}$*, then the time-ordered integral* $\mathbf{Q}_{\mathbf{I}}[t,a] = \int_a^t \mathcal{A}_{\mathbf{I}}(s)ds$*, where* $\mathcal{A}_{\mathbf{I}}(t) = \bar{\mathbf{U}}_0^\sigma[a,t]\mathcal{A}_1(t)\bar{\mathbf{U}}_0^\sigma[t,a]$*, generates a* C_0*-unitary group on* $\mathcal{FD}_\otimes^2$*, and*

$$\exp\{(-i/\hbar)\mathbf{Q}_{\mathbf{I}}^n[t,a]\} \to \exp\{(-i/\hbar)\mathbf{Q}_{\mathbf{I}}[t,a]\},$$

where $\mathbf{Q}_{\mathbf{I}}^{n}[t,a] = \int_a^t \mathcal{A}_{\mathbf{I}}^{n}(s)ds$, *and:*

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = \mathcal{A}_{\mathbf{I}}(t)\Psi(t), \ \Psi(a) = \Phi.$$

Proof. The result follows from an application of Theorems 62 and 63. □

Definition 64. *The evolution operator* $\mathbf{U}^{w}[t,a] = exp\{w\mathbf{Q}[t,a]\}$ *is said to be asymptotic in the sense of Poincaré if, for each* n *and each* $\Phi_a \in D\left[(\mathbf{Q}[t,a])^{n+1}\right]$, *we have*

$$\lim_{w\to 0} w^{-(n+1)}\left\{\mathbf{U}^{w}[t,a] - \sum_{k=1}^{n}\frac{(w\mathbf{Q}[t,a])^{k}}{k!}\right\}\Phi_a = \frac{\mathbf{Q}[t,a]^{n+1}}{(n+1)!}\Phi_a. \tag{5.1}$$

This is the operator version of an asymptotic expansion in the classical sense, but $\mathbf{Q}[t,a]$ *is now an unbounded operator.*

Theorem 65. *Suppose that* $\mathbf{Q}[t,a]$ *generates a contraction* C_0*-semigroup on* $\mathcal{FD}_{\otimes}^{2}$ *for each* $t \in I$. *Then:*

The operator $\mathbf{U}^{w}[t,a] = exp\{w\mathbf{Q}[t,a]\}$ *is asymptotic in the sense of Poincaré.*

For each n *and each* $\Phi_a \in D\left[(\mathbf{Q}[t,a])^{n+1}\right]$, *we have*

$$\begin{aligned}\Phi(t) = \Phi_a + \sum_{k=1}^{n} w^{k}\int_a^t ds_1\int_a^{s_1} ds_2\cdots\int_a^{s_{k-1}} ds_k\mathcal{A}(s_1)\mathcal{A}(s_2)\cdots\mathcal{A}(s_k)\Phi_a \\ + \int_0^w (w-\xi)^n d\xi\int_a^t ds_1\int_a^{s_1} ds_2\cdots\int_a^{s_n} ds_{n+1}\mathcal{A}(s_1)\mathcal{A}(s_2)\cdots\mathcal{A}(s_{n+1})\mathbf{U}^{\xi}[s_{n+1},a]\Phi_a,\end{aligned} \tag{5.2}$$

where $\Phi(t) = \mathbf{U}^{w}[t,a]\Phi_a$.

Remark 66. *The above case includes all generators of C_0-unitary groups. Thus, the theorem provides a precise formulation and proof of Dyson's second conjecture for quantum electrodynamics, that, in general, we can only expect the expansion to be asymptotic. Actually, we prove more in that we produce the remainder term, so that the above perturbation expansion is exact for all finite n.*

Proof. From Theorem 58, we have

$$\mathbf{U}^w[t,a]\Phi = \left\{ \sum_{k=0}^{n} \frac{(w\mathbf{Q}[t,a])^n}{n!} + \frac{1}{n!} \int_0^w (w-\xi)^n \mathbf{Q}[t,a]^{n+1} \mathbf{U}^\xi[t,a] d\xi \right\} \Phi,$$

so that

$$w^{-(n+1)} \left\{ \mathbf{U}^w[t,a]\Phi_a - \sum_{k=0}^{n} \frac{(w\mathbf{Q}[t,a])^k}{k!} \Phi_a \right\} = +\frac{(n+1)}{(n+1)!} w^{-(n+1)} \int_0^w (w-\xi)^n d\xi \mathbf{U}^\xi[t,a] \mathbf{Q}[t,a]^{n+1} \Phi_a.$$

Replace the right hand side by

$$\mathbf{I} = \frac{(n+1)}{(n+1)!} w^{-(n+1)} \int_0^w (w-\xi)^n d\xi \left\{ \mathbf{U}_z^\xi[t,a] + \left[\mathbf{U}^\xi[t,a] - \mathbf{U}_z^\xi[t,a] \right] \right\} \mathbf{Q}[t,a]^{n+1} \Phi_a$$

$$= \mathbf{I}_{1,z} + \mathbf{I}_{2,z},$$

where

$$\mathbf{I}_{1,z} = \frac{(n+1)}{(n+1)!} w^{-(n+1)} \int_0^w (w-\xi)^n d\xi \mathbf{U}_z^\xi[t,a] \mathbf{Q}[t,a]^{n+1} \Phi_a,$$

and

$$\mathbf{I}_{2,\mathrm{z}} = \frac{(n+1)}{(n+1)!} w^{-(n+1)} \int_0^w (w-\xi)^n d\xi \left[\mathbf{U}^\xi[t,a] - \mathbf{U}_{\mathrm{z}}^\xi[t,a] \right] \mathbf{Q}[t,a]^{n+1} \Phi_a.$$

From the proof of Theorem 58, we see that $\lim_{z\to\infty}\mathbf{I}_{2,z}=0$. Let $\varepsilon>0$ be given and choose Z such that $\mathrm{z}>\mathrm{Z}\Rightarrow\left\|\mathbf{I}_{2,\mathrm{z}}\right\|<\varepsilon$. Now, use

$$\mathbf{U}_{\mathrm{z}}^{\xi}[t,a]=\mathrm{I}_{\otimes}+\sum\nolimits_{k=1}^{\infty}\frac{\xi^{k}\mathbf{Q}_{\mathrm{z}}^{\mathrm{k}}[t,a]}{k!}$$

for the first term to get that

$$\mathbf{I}_{1,z}=\frac{(n+1)}{(n+1)!}w^{-(n+1)}\int\limits_{0}^{w}(w-\xi)^{n}d\xi\left\{\mathrm{I}_{\otimes}+\sum\nolimits_{k=1}^{\infty}\frac{\xi^{k}\mathbf{Q}_{z}^{k}[t,a]}{k!}\right\}\mathbf{Q}[t,a]^{n+1}\Phi_{a}.$$

If we compute the elementary integrals, we get

$$\begin{aligned}\mathbf{I}_{1,\mathrm{z}}=&\frac{1}{(n+1)!}\mathbf{Q}[t,a]^{n+1}\Phi_{a}\\&+\sum\nolimits_{\mathrm{k}\,=\,1}^{\infty}\frac{1}{k!n!}\left\{\sum\nolimits_{l=1}^{n}\begin{pmatrix}n\\ l\end{pmatrix}\frac{w^{k}}{(n+k+1-l)}\right\}\mathbf{Q}_{\mathrm{z}}^{k}[t,a]\mathbf{Q}[t,a]^{n+1}\Phi_{a}\end{aligned}$$

then

$$\begin{aligned}&\left\|\mathbf{I}-\frac{1}{(n+1)!}\mathbf{Q}[t,a]^{n+1}\Phi_{a}\right\|<\\&\left\|\sum\nolimits_{\mathrm{k}\,=\,1}^{\infty}\frac{1}{k!n!}\left\{\sum\nolimits_{l=1}^{n}\begin{pmatrix}n\\ l\end{pmatrix}\frac{w^{k}}{(n+k+1-l)}\right\}\mathbf{Q}_{\mathrm{z}}^{k}[t,a]\mathbf{Q}[t,a]^{n+1}\Phi_{a}\right\|+\varepsilon.\end{aligned}$$

Now let $w\to 0$ to get

$$\left\|\mathbf{I}-\frac{1}{(n+1)!}\mathbf{Q}[t,a]^{n+1}\Phi_{a}\right\|<\varepsilon.$$

Since ε is arbitrary, $\mathbf{U}[t,a]=\exp\left\{\mathbf{Q}[t,a]\right\}$ is asymptotic in the sense of Poincaré.

To prove (5.2), let $\Phi_a \in D\left[(\mathbf{Q}[t,a])^{n+1}\right]$ for each k $\leqslant$ n + 1, and use the fact that (Dollard and Friedman [DOF])

$$(5.3)\qquad \begin{aligned} (\mathbf{Q}_{\mathrm{z}}[t,a])^k\,\Phi_a &= \left(\int\limits_a^t \mathcal{A}_{\mathrm{z}}(s)ds\right)^k \Phi_a \\ &= (k!)\int\limits_a^t ds_1 \int\limits_a^{s_1} ds_2 \cdots \int\limits_a^{s_{k-1}} ds_n \mathcal{A}_{\mathrm{z}}(s_1)\mathcal{A}_{\mathrm{z}}(s_2)\cdots\mathcal{A}_{\mathrm{z}}(s_k)\Phi_a. \end{aligned}$$

Letting $z \to \infty$ gives the result. □

There are special cases in which the perturbation series may actually converge to the solution. It is known that, if $A_0(t)$ is a nonnegative self-adjoint operator on $\mathcal{H}$, then $\exp\{-\tau A_0(t)\}$ is an analytic C_0-contraction semigroup for $\operatorname{Re}\tau > 0$ (see Kato [KA], pg. 491). More generally, if $\Delta = \{z \in \mathbf{C} : \varphi_1 < \arg z < \varphi_2,\ \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$, suppose that $T(z)$ is a bounded linear operator on $\mathcal{H}$.

Definition 67. *The family $T(w)$ is said to be an analytic semigroup on $\mathcal{H}$, for $w \in \Delta$, if*

(1) *$T(w)f$ is an analytic function of $w \in \Delta$ for each f in $\mathcal{H}$.*

(2) *$T(0) = I$ and $\lim_{w\to 0} T(w)f = f$ for every $f \in \mathcal{H}$.*

(3) *$T(w_1 + w_2) = T(w_1)T(w_2)$ for $w_1,\ w_2 \in \Delta$.*

For a proof of the next theorem, see Pazy [PZ], page 61.

Theorem 68. *Let A_0 be a closed densely defined linear operator defined on $\mathcal{H}$, satisfying:*

(1) *For some* $0 < \delta < \pi/2$,

$$\rho(A_0) \supset \Sigma_\delta = \{\lambda \,:\, |\arg \lambda| < \pi/2 + \delta\} \cup \{0\}.$$

(2) *There is a constant* M *such that:*

$$\|R(\lambda \,:\, A_0)\| \leqslant M/|\lambda|$$

for $\lambda \in \Sigma_\delta$, $\lambda \neq 0$.

Then A_0 *is the infinitesimal generator of a uniformly bounded analytic semigroup* $T(w)$, *for* $w \in \bar{\Delta}_{\delta'} = \{w \,:\, |\arg w| \leqslant \delta' < \delta\}$. *Furthermore, for* $s > 0$ *and* $|w - s| \leqslant Cs$ *for some constant* C,

$$T(w+s) = T(s) + \sum\nolimits_{n=1}^{\infty} (w^n/n!) T^{(n)}(s),$$

and the series converges uniformly.

Theorem 69. *Let* $\mathbf{Q}_0[t,a] = \int\limits_a^t \mathcal{A}_0(s)ds$ *and* $\mathbf{Q}_1[t,a] = \int\limits_a^t \mathcal{A}_1(s)ds$ *be nonnegative selfadjoint generators of analytic* C_0*-contraction semigroups for* $t \in (a,b]$. *Suppose* $D(\mathbf{Q}_1[t,a]) \supseteq D(\mathbf{Q}_0[t,a])$ *and there are positive constants* α, β *such that*

$$\|\mathbf{Q}_1[t,a]\Phi\|_{\otimes} \leqslant \alpha \|\mathbf{Q}_0[t,a]\Phi\|_{\otimes} + \beta \|\Phi\|_{\otimes}, \; \Phi \in D(\mathbf{Q}_0[t,a]). \tag{5.4}$$

(1) *Then* $\mathbf{Q}[t,a] = \mathbf{Q}_0[t,a] + \mathbf{Q}_1[t,a]$ *and* $\mathcal{A}_{\mathbf{I}}(t) = \bar{\mathbf{U}}_0[a,t]\mathcal{A}_1(t)\bar{\mathbf{U}}_0[t,a]$ *both generate analytic* C_0*-contraction semigroups and, for* w *small enough, we have*

(2) *For each k and each* $\Phi_a \in D\left[(\mathbf{Q_I}[t,a])^{k+1}\right]$,

$$\mathbf{U}_{\mathbf{I}}^{w}[t,a]\Phi_a = \Phi_a + \sum_{l=1}^{k} w^l \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{k-1}} ds_k \mathcal{A}_{\mathbf{I}}(s_1)\mathcal{A}_{\mathbf{I}}(s_2)\cdots\mathcal{A}_{\mathbf{I}}(s_k)\Phi_a$$

$$+ \int_0^w (w-\xi)^k d\xi \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_k} ds_{k+1} \mathcal{A}_{\mathbf{I}}(s_1)\mathcal{A}_{\mathbf{I}}(s_2)\cdots\mathcal{A}_{\mathbf{I}}(s_{k+1})\mathbf{U}_{\mathbf{I}}^{\xi}[s_{k+1},a]\Phi_a.$$

(3) *If* $\Phi_a \in \cap_{k\geqslant 1} D\left[(\mathbf{Q_I}[t,a])^{k}\right]$, *we have*

$$\mathbf{U}_{\mathbf{I}}^{w}[t,a]\Phi_a = \Phi_a + \sum_{k=1}^{\infty} w^l \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{k-1}} ds_k \mathcal{A}_{\mathbf{I}}(s_1)\mathcal{A}_{\mathbf{I}}(s_2)\cdots\mathcal{A}_{\mathbf{I}}(s_k)\Phi_a.$$

Proof. To prove (1), use the fact that $\mathbf{Q}_0[t,a]$ generates an analytic C_0-contraction semigroup to find a sector Σ in the complex plane, with $\rho(\mathbf{Q}_0[t,a]) \supset \Sigma$ ($\Sigma = \{\lambda : \ |\arg\lambda| < \pi/2 + \delta'\}$, for some $\delta' > 0$), and for $\lambda \in \Sigma$,

$$\|R(\lambda : \mathbf{Q}_0[t,a])\|_{\otimes} \leqslant |\lambda|^{-1}.$$

From (5.4), $\mathbf{Q}_1[t,a]R(\lambda : \mathbf{Q}_0[t,a])$ is a bounded operator and:

$$\|\mathbf{Q}_1[t,a]R(\lambda : \mathbf{Q}_0[t,a])\Phi\|_{\otimes} \leqslant \alpha \|\mathbf{Q}_0[t,a]R(\lambda : \mathbf{Q}_0[t,a])\Phi\|_{\otimes} + \beta \|R(\lambda : \mathbf{Q}_0[t,a])\Phi\|_{\otimes}$$

$$\leqslant \alpha \|[R(\lambda : \mathbf{Q}_0[t,a]) - \mathbf{I}]\,\Phi\|_{\otimes} + \beta\,|\lambda|^{-1}\,\|\Phi\|_{\otimes}$$

$$\leqslant 2\alpha\,\|\Phi\|_{\otimes} + \beta\,|\lambda|^{-1}\,\|\Phi\|_{\otimes}$$

Thus, if we set $\alpha = 1/4$, and $|\lambda| > 2\beta$, we have

$$\|\mathbf{Q}_1[t,a]R(\lambda : \mathbf{Q}_0[t,a])\|_{\otimes} < 1$$

and it follows that the operator

$$\mathbf{I} - \mathbf{Q}_1[t,a]R(\lambda : \mathbf{Q}_0[t,a])$$

is invertible. Now it is easy to see that:

$$(\lambda\mathbf{I} - (\mathbf{Q}_0[t,a] + \mathbf{Q}_1[t,a]))^{-1} = R(\,\lambda\,:\,\mathbf{Q}_0[t,a])\,(\mathbf{I} - \mathbf{Q}_1[t,a]R(\,\lambda\,:\,\mathbf{Q}_0[t,a]))^{-1}\,.$$

It follows that, using $|\lambda| > 2\beta$, with $|\arg\lambda| < \pi/2 + \delta''$ for some $\delta'' > 0$, and the fact that $\mathbf{Q}_0[t,a]$ and $\mathbf{Q}_1[t,a]$ are nonnegative generators, we get that

$$\|R(\,\lambda\,:\,\mathbf{Q}_0[t,a] + \mathbf{Q}_1[t,a])\|_{\otimes} \leqslant |\lambda|^{-1}\,.$$

Thus $\mathbf{Q}_0[t,a]+\mathbf{Q}_1[t,a]$ generates an analytic C_0-contraction semigroup. The proof of (2) follows from Theorem 66, and that of (3) follows from Theorem 69. □

There are also cases where the series may diverge, but still respond to some summability method. This phenomenon is well known in classical analysis. In field theory, things can be much more complicated. The book by Glimm and Jaffe [GJ] has a good discussion.

6. **Path Integrals II: Sum Over Paths**

In his book Feynman suggested that the operator calculus was more general than the path integral (see Feynman and Hibbs [FH], pg. 355-6). In this section, we first construct (what we call) the experimental evolution operator. This allows us to rewrite our theory as a sum over paths. We use a general argument so that the ideas apply to almost all cases. Assume that the family $\{\tau_1, \tau_2, \cdots, \tau_n\}$ represents the time positions of n possible measurements of a general system trajectory, as appears on a film of system

history. We assume that information is available beginning at time $T = 0$ and ends at time $T = t$. Define $\mathbf{Q}_E[\tau_1, \tau_2, \cdots, \tau_n]$ by

$$\mathbf{Q}_E[\tau_1, \tau_2, \cdots, \tau_n] = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E[\tau_j, s]\mathcal{A}(s)ds. \tag{6.1}$$

Here, $t_0 = \tau_0 = 0$, $t_j = (1/2)[\tau_j + \tau_{j+1}]$ (for $1 \leqslant j \leqslant n$), and $E[\tau_j, s]$ is the exchange operator. The effect of $E[\tau_j, s]$ is to concentrate all information contained in $[t_{j-1}, t_j]$ at τ_j, the mid-point of the time interval around τ_j relative to τ_{j-1} and τ_{j+1}. We can rewrite $\mathbf{Q}_E[\tau_1, \tau_2, \cdots, \tau_n]$ as

$$\mathbf{Q}_E[\tau_1, \tau_2, \cdots, \tau_n] = \sum_{j=1}^{n} \Delta t_j \left[\frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} E[\tau_j, s]\mathcal{A}(s)ds\right]. \tag{6.2}$$

Thus, we have an average over each adjacent interval, with information concentrated at the mid-point. The evolution operator is given by

$$U[\tau_1, \tau_2, \cdots, \tau_n] = \exp\left\{\sum_{j=1}^{n} \Delta t_j \left[\frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} E[\tau_j, s]\mathcal{A}(s)ds\right]\right\}.$$

For $\Phi \in \mathcal{FD}_{\otimes}^2$, we define the function $\mathbf{U}[N(t), 0]\Phi$ by:

$$\mathbf{U}[N(t), 0]\Phi = U[\tau_1, \tau_2, \cdots, \tau_n]\Phi. \tag{6.3}$$

$\mathbf{U}[N(t), 0]\Phi$ is a $\mathcal{FD}_{\otimes}^2$-valued random variable, which represents the distribution of the number of measurements, $N(t)$, that are possible up to time t. In order to relate $\mathbf{U}[N(t), 0]\Phi$ to actual experimental results, we must compute its expected value. Let λ^{-1} denote the smallest time interval in which a measurement can be made, and define $\bar{\mathbf{U}}_\lambda[t, 0]\Phi$ by:

$$\bar{\mathbf{U}}_\lambda[t, 0]\Phi = \mathcal{E}\left[\mathbf{U}[N(t), 0]\Phi\right] = \sum_{n=0}^{\infty} \mathcal{E}\left\{\mathbf{U}[N(t), 0]\Phi \,|N(t) = n\right\} \Pr ob\left[N(t) = n\right].$$

We make the natural assumption that: (See Gill and Zachary [GZ])

$$\Pr ob\,[N(t) = n] = (n!)^{-1}\,(\lambda t)^n \exp\{-\lambda t\}.$$

The expected value-integral is of theoretical use and is not easy to compute. Since we are only interested in what happens when $\lambda \to \infty$, and as the mean number of possible measurements up to time t is λt, we can take $\tau_j = (jt/n)$, $1 \leqslant j \leqslant n$, ($\Delta t_j = t/n$ for each n). We can now replace $\bar{\mathbf{U}}_n[t{,}0]\Phi$ by $\mathbf{U}_n[t{,}0]\Phi$, and with this understanding, we continue to use τ_j, so that

$$\mathbf{U}_n[t{,}0]\Phi = \exp\left\{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E[\tau_j, s]\mathcal{A}(s)ds\right\}\Phi. \tag{6.4}$$

We define our experimental evolution operator $\mathbf{U}_\lambda[t,0]\Phi$ by

$$\mathbf{U}_\lambda[t,0]\Phi = \sum_{n=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^n}{n!} \exp\{-\lambda t\}\mathbf{U}_n[t{,}0]\Phi. \tag{6.5}$$

We now have the following result, which is a consequence of the fact that Borel summability is regular.

Theorem 70. *Assume that the conditions for Theorem 51 are satisfied. Then*

$$\lim_{\lambda\to\infty} \bar{\mathbf{U}}_\lambda[t,0]\Phi = \lim_{\lambda\to\infty} \mathbf{U}_\lambda[t,0]\Phi = \mathbf{U}[t,0]\Phi. \tag{6.6}$$

Since $\lambda \to \infty \Rightarrow \lambda^{-1} \to 0$, this means that the average time between measurements is zero (in the limit) so that we get a continuous path. It should be observed that this continuous path arises from averaging the sum over

an infinite number of (discrete) paths. The first term in (6.5) corresponds to the path of a system that created no information (i.e., the film is blank). This event has probability $\exp\{-\lambda t\}$ (which approaches zero as $\lambda \to \infty$). The n-th term corresponds to the path that creates n possible masurements, (with probability $[(\lambda t)^n/n!]\exp\{-\lambda t\}$) etc.

Let $U[t,a]$ be an evolution operator on $L^2[\mathbf{R}^3]$, with time-dependent generator $A(t)$, which has a kernel $\mathbf{K}[\mathbf{x}(t),t\,;\,\mathbf{x}(s),s]$ such that:

$$\mathbf{K}\left[\mathbf{x}(t),\,t;\,\mathbf{x}(s),\,s\right] = \int_{\mathbf{R}^3} \mathbf{K}\left[\mathbf{x}(t),\,t;\,d\mathbf{x}(\tau),\,\tau\right]\mathbf{K}\left[\mathbf{x}(\tau),\,\tau;\,\mathbf{x}(s),\,s\right],$$

$$U[t,s]\varphi(s) = \int_{\mathbf{R}^3} \mathbf{K}\left[\mathbf{x}(t),\,t;\,d\mathbf{x}(s),\,s\right]\varphi(s).$$

Now let $\mathcal{H} = \mathcal{KS}[\mathbf{R}^3] \supset L^2[\mathbf{R}^3]$ in the construction of $\mathcal{FD}^2_\otimes \subset \mathcal{H}^2_\otimes$, let $\mathbf{U}[t,s]$ be the corresponding time-ordered version, with kernel $\mathbb{K}_{\mathbf{f}}\left[\mathbf{x}(t),\,t;\,\mathbf{x}(s),\,s\right]$. Since $\mathbf{U}[t,\tau]\mathbf{U}[\tau,s] = \mathbf{U}[t,s]$, we have:

$$\mathbb{K}_{\mathbf{f}}\left[\mathbf{x}(t),\,t;\,\mathbf{x}(s),\,s\right] = \int_{\mathbf{R}^3} \mathbb{K}_{\mathbf{f}}\left[\mathbf{x}(t),\,t;\,d\mathbf{x}(\tau),\,\tau\right]\mathbb{K}_{\mathbf{f}}\left[\mathbf{x}(\tau),\,\tau;\,\mathbf{x}(s),\,s\right].$$

From our sum over paths representation for $\mathbf{U}[t,s]$, we have:

$$\begin{aligned}\mathbf{U}[t,s]\Phi(s) &= \lim_{\lambda\to\infty}\mathbf{U}_\lambda[t,s]\Phi(s)\\ &= \lim\nolimits_{\lambda\to\infty} \mathrm{e}^{-\lambda(t-s)} \sum_{k=0}^{[\![\lambda(t-s)]\!]} \frac{\left[\lambda\left(t-s\right)\right]^k}{k!}\mathbf{U}_k[t,s]\Phi(s),\end{aligned}$$

where

$$\mathbf{U}_k[t,s]\Phi(s) = \exp\left\{(-i/\hbar)\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\mathbf{E}[(j/\lambda),\tau]\mathcal{A}(\tau)d\tau\right\}\Phi(s).$$

As in Section 1, we define $\mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(\mathrm{s})]$ by:

$$\int_{\mathbf{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(\mathrm{s})]$$
$$=: \mathrm{e}^{-\lambda(t-s)} \sum_{k=0}^{[\![\lambda(t-s)]\!]} \frac{[\lambda(t-s)]^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbf{R}^3} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(t_j)\ ;\ d\mathbf{x}(t_{j-1}), t_{j-1}]|^{(j/\lambda)} \right\},$$

where $[\![\lambda(t-s)]\!]$, the greatest integer in $\lambda(t-s)$, and $|^{(j/\lambda)}$ denotes the fact that the integration is performed in time slot (j/λ).

Definition 71. *We define the Feynman path integral associated with* $\mathbf{U}[t,s]$ *by:*

$$\mathbf{U}[t,s] = \int_{\mathbf{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)] = \lim_{\lambda\to\infty} \int_{\mathbf{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(s)].$$

Theorem 72. *For the time-ordered theory, whenever a kernel exists, we have that:*

$$\lim_{\lambda\to\infty} \mathbf{U}_\lambda[t,s]\Phi(s) = \mathbf{U}[t,s]\Phi(s) = \int_{\mathbf{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)]\Phi[\mathbf{x}(s)],$$

and the limit is independent of the space of continuous functions.

. Let us assume that $A_0(t)$ and $A_1(t)$ are strongly continuous generators of C_0-contraction semigroups for each $t \in E = [a,b]$ and, let $\mathcal{A}_{1,\rho}(t) = \rho\mathcal{A}_1(t)\mathbf{R}(\rho, \mathcal{A}_1(t))$ be the Yosida approximator for the time-ordered version of $A_1(t)$. Define $\mathbf{U}^\rho[t,a]$ and $\mathbf{U}^0[t,a]$ by:

$$\mathbf{U}^\rho[t,a] = \exp\{(-i/\hbar)\int_a^t [\mathcal{A}_0(s) + \mathcal{A}_{1,\rho}(s)]ds\},$$

$$\mathbf{U}^0[t,a] = \exp\{(-i/\hbar)\int_a^t \mathcal{A}_0(s)ds\}.$$

Since $\mathcal{A}_{1,\rho}(s)$ is bounded, $\mathcal{A}_0(s) + \mathcal{A}_{1,\rho}(s)$ is a generator of a C_0-contraction semigroup for $s \in E$ and finite ρ. Now assume that $\mathbf{U}^0[t,a]$ has an associated kernel, so that $\mathbf{U}^0[t,a] = \int_{\mathbf{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(a)]$. We now have the following general result, which is independent of the space of continuous functions.

Theorem 73. *(Feynman-Kac)* If $\mathcal{A}_0(s) \oplus \mathcal{A}_1(s)$ is a generator of a C_0-contraction semigroup, then*

$$\lim_{\rho\to\infty} \mathbf{U}^\rho[t,a]\Phi(a) = \mathbf{U}[t,a]\Phi(a)$$
$$= \int_{\mathbf{R}^{3[t,a]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(a)] \exp\{(-i/\hbar) \int\limits_a^\tau \mathcal{A}_1(s)ds]\}\Phi[\mathbf{x}(a)].$$

Proof. The fact that $\mathbf{U}^\rho[t,a]\Phi(a) \to \mathbf{U}[t,a]\Phi(a)$, is clear. To prove that

$$\mathbf{U}[t,a]\Phi(a) = \int_{\mathbf{R}^{3[t,a]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(a)] \exp\{(-i/\hbar)\int_a^t \mathcal{A}_1(s)ds\},$$

first note that since the time-ordered integral exists and we are only interested in the limit, we can write, for each k

$$U_k^\rho[t,a]\Phi(a) = \exp\left\{(-i/\hbar)\sum\nolimits_{j=1}^k \int_{t_{j-1}}^{t_j} \left[\mathbf{E}[\tau_j,s]\mathcal{A}_0(s) + \mathbf{E}[\tau_j',s]\mathcal{A}_{1,\rho}(s)\right] ds\right\}$$

where τ_j and τ_j' are distinct points in the interval (t_{j-1}, t_j). Thus, we can also write $U_k^\rho[t,a]\Phi(a)$ as

$$\begin{aligned}
&\mathbf{U}_k^\rho[t,a]\Phi(a) \\
&= \exp\left\{(-i/\hbar)\sum\nolimits_{j=1}^k \int_{t_{j-1}}^{t_j} \mathbf{E}[\tau_j, s]\mathcal{A}_0(s)ds\right\} \exp\left\{(-i/\hbar)\sum\nolimits_{j=1}^k \mathbf{E}[\tau_j', s]\mathcal{A}_{1,\rho}(s)ds\right\} \\
&= \prod\nolimits_{j=1}^k \exp\left\{(-i/\hbar)\int_{t_{j-1}}^{t_j} \mathbf{E}[\tau_j, s]\mathcal{A}_0(s)ds\right\} \exp\left\{(-i/\hbar)\sum\nolimits_{j=1}^k \mathbf{E}[\tau_j', s]\mathcal{A}_{1,\rho}(s)ds\right\} \\
&= \prod\nolimits_{j=1}^k \int_{R^3} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(t_j); t_{j-1}, d\mathbf{x}(t_{j-1})]\ |^{\tau_j} \exp\left\{(-i/\hbar)\sum\nolimits_{j=1}^k \mathbf{E}[\tau_j', s]\mathcal{A}_{1,\rho}(s)ds\right\}.
\end{aligned}$$

If we put this in our experimental evolution operator $\mathbf{U}_\lambda^\rho[t,a]\Phi(a)$ and compute the limit, we have:

$$\begin{aligned}
&\mathbf{U}^\rho[t,a]\Phi(a) \\
&= \int_{\mathbf{R}^{3[t,a]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(t); \mathbf{x}(a)] \exp\left\{(-i/\hbar)\int_a^t \mathcal{A}_{1,\rho}(s)ds\right\}\Phi(a).
\end{aligned}$$

Since the limit as $\rho \to \infty$ on the left exists, it defines the limit on the right. $\square$

7. Discussion

The reader may have noticed that there is no discussion of the various Trotter-Kato product type theorems, which have played an important role in the applications of semigroup theory. This theory identifies conditions under which the sum of two or more semigroup generators is a generator and as such, carries over to the time-ordered setting without any changes in the basic results.

The question of external forces requires discussion of the inhomogeneous problem. Since the inhomogeneous problem is a special case of the semilinear problem, we provide a few remarks in that direction. Since all of the standard results go through as in the conventional approach, we content ourselves with a brief description of a typical case. Without loss in generality, we assume $\mathcal{H}$ has our standard basis. With the conditions for the parabolic or hyperbolic problem in force, the typical semilinear problem can be represented on $\mathcal{H}$ as:

$$\frac{\partial u(t)}{\partial t} = A(t)u(t) + f(t, u(t)),\ u(a) = u_a. \tag{7.1}$$

We assume that f is continuously differentiable with $u_0 \in \mathcal{H}$ in the parabolic or $u_0 \in D$, the common dense domain, in the hyperbolic case. These conditions are sufficient for $u(t)$ to be a classical solution (see Pazy [PZ], pg. 187). The function f has the representation $f(t, u(t)) = \sum_{k=1}^{\infty} f_k(t)e^k$ in $\mathcal{H}$. The corresponding function $\mathbf{f}$, in $\mathcal{FD}_{\otimes}^2$, has the representation $\mathbf{f}(t, \mathbf{u}(t)) = \sum_{k=1}^{\infty} f_k(t)E^k$, where $\mathbf{u}(t)$ is a classical solution to the time-ordered problem:

$$\frac{\partial \mathbf{u}(t)}{\partial t} = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{f}(t, \mathbf{u}(t)),\ \mathbf{u}(a) = \mathbf{u}_a. \tag{7.2}$$

This function $\mathbf{u}(t)$ also satisfies the integral equation (time-ordered mild solution):

$$\mathbf{u}(t) = \mathbf{U}(t, a)\mathbf{u}_a + \int_a^t \mathbf{U}(t, s)\mathbf{f}(s, \mathbf{u}(s))ds.$$

If f does not depend on $u(t)$, we get the standard linear inhomogeneous problem. It follows that all the basic results (and proofs) go though for the semilinear and linear inhomogeneous problem in the time-ordered case. Similar statements apply to the problem of asymptotic behavior of solutions (e.g., dynamical systems, attractors, etc).

The general nonlinear problem requires a different approach, that depends on a new theory of nonlinear operator algebras, which we call $\mathbf{S}^*$-algebras. This will be discuss at a later time, however the theory has recently been (indirectly) used to construct a sufficiency class of functions for global (in time) solutions to the 3D-Navier-Stokes equations [GZ1]. The corresponding linear theory can be found in [GZ2] .

Conclusion

In this paper we have shown how to construct a natural representation space for Feynman's time-ordered operator calculus. This space allows us to construct the time-ordered integral and evolution operator (propagator) under the weakest known conditions. We have constructed a new Hilbert space that contains the Feynman kernel and the delta function as norm bounded elements, and shown that on this space, we can rigorously construct the path integral in the manner originally intended by Feynman. We have extended the path integral to very general interactions and provided

a substantial generalization of the Feynman-Kac formula. We have also developed a general theory for perturbations and shown that all time-ordered evolution operators are asymptotic in the operator-valued sense of Poincaré.

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